12.4 Tangent and Normal Vectors Cont'd

Tangential component of acceleration

$$a_T = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$$

Normal component of acceleration

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$$

Ex: Given $\vec{r}(t) = [t, \frac{t^2}{2}, \frac{t^3}{3}]$
Find a_T and a_N.

$$\vec{v} = [1, t, t^2]$$

$$\vec{a} = [0, 1, 2t]$$

$$a_T = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$$

$$= \frac{t + 2t^3}{\sqrt{1 + t^2 + t^4}}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix}$$

$$= \vec{i}\,(2t^2 - t^2) - \vec{j}\,(\ 2t\ ) + \vec{k}\,(\ 1\ )$$

$$= [t^2, -2t, 1]$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{t^4 + 4t^2 + 1}$$

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$$

$$= \frac{\sqrt{t^4 + 4t^2 + 1}}{\sqrt{t^4 + t^2 + 1}}$$

<u>Ex</u>: Let $\vec{r} = [e^t, e^t \cos t, e^t \sin t]$.

Find a_T and a_N, evaluated at $t = 0$.

$$\vec{v} = [e^t, -e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t]$$

$$\vec{a} = [e^t, \cancel{-e^t \cos t} \underbrace{- e^t \sin t - e^t \sin t}_{-2e^t \sin t} + \cancel{e^t \cos t}, \cancel{-e^t \sin t} + \underbrace{e^t \cos t + e^t \cos t}_{2e^t \cos t} + \cancel{e^t \sin t}]$$

$$\vec{v}(0) = [1, 1, 1]$$

$$\vec{a}(0) = [1, 0, 2]$$

$$a_T(0) = \frac{\vec{v}(0) \cdot \vec{a}(0)}{\|\vec{v}(0)\|}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3}$$

$$a_N(0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{\|\vec{v}(0)\|}$$

$$\boxed{\begin{array}{l} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\ = \vec{i}(2) - \vec{j}(1) + \vec{k}(-1) \\ = [2, -1, -1] \end{array}}$$

$$= \frac{\sqrt{6}}{\sqrt{3}}$$

$$= \sqrt{2}$$

12.5 Arc Length

$t=b$

$t=a$

$s =$ arc length

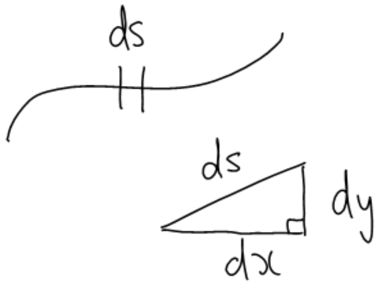
$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{(in 2D)}$$

or

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \text{(in 3D)}$$

$$S = \int_a^b \| \vec{v}(t) \| \, dt \qquad \text{(in 2D or 3D)}$$

Why?



$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) dt^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \| \vec{v}(t) \| \, dt$$

Ex: Find the arc length of
$\vec{r}(t) = [\cos 3t, \sin 3t, 2t]$ on $0 \leq t \leq 4\pi$.

$$\vec{v} = [-3\sin 3t, 3\cos 3t, 2]$$

$$\| \vec{v} \| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 4}$$

$$= \sqrt{9\,\cancel{(\sin^2 3t + \cos^2 3t)} + 4}$$

$$= \sqrt{13}$$

$$s = \int_a^b \|\vec{v}\| \, dt$$

$$= \int_0^{4\pi} \sqrt{13} \, dt$$

$$= \sqrt{13} \, t \Big|_0^{4\pi}$$

$$= 4\pi\sqrt{13}$$

<u>Ex</u>: Find the arc length of

$\vec{r} = [2t^2+1, \; t^3+5]$ on $0 \le t \le 1$.

$$\vec{v} = [4t, \; 3t^2]$$

$$\|\vec{v}\| = \sqrt{16t^2 + 9t^4}$$

$$= \sqrt{t^2(16+9t^2)}$$

$$= |t| \sqrt{16+9t^2}$$

$$= t\sqrt{16+9t^2} \qquad (0 \le t \le 1)$$

$$s = \int_0^1 t\sqrt{16+9t^2} \, dt$$

$$\vdots$$