12.2 Derivatives and Integrals of Vector-Valued Functions Cont'd

Ex: $\vec{r}(t) = [\,t^2,\ 7t,\ t^3\,]$

Find $\vec{r}\,'(t) \cdot \vec{r}\,''(t)$

and $\vec{r}\,'(t) \times \vec{r}\,''(t)$

$$\vec{r}\,'(t) = [\,2t,\ 7,\ 3t^2\,]$$
$$\vec{r}\,''(t) = [\,2,\ 0,\ 6t\,]$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 2t(2) + 7(0) + 3t^2(6t)$$
$$= 4t + 18t^3$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t & 7 & 3t^2 \\ 2 & 0 & 6t \end{vmatrix}$$

$$= \vec{\imath}\,(42t) - \vec{\jmath}\,(12t^2 - 6t^2) + \vec{k}\,(-14)$$

$$= 42t\,\vec{\imath} - 6t^2\,\vec{\jmath} - 14\vec{k}$$

or $[\,42t,\ -6t^2,\ -14\,]$

FACT

If $\vec{r}(t) = [\,x(t),\ y(t),\ z(t)\,]$

then $\int \vec{r}(t)\,dt = [\,\int x(t)\,dt,\ \int y(t)\,dt,\ \int z(t)\,dt\,]$

Ex: Find $\vec{r}(t)$ if

$$\vec{r}\,'(t) = [3t^2, 2t, 6e^{2t}]$$

and $\vec{r}(0) = [3, 2, 8]$.

$$\vec{r}(t) = [t^3 + C_1, t^2 + C_2, 3e^{2t} + C_3]$$

or $\vec{r}(t) = [t^3, t^2, 3e^{2t}] + \vec{C}$

Sub $t=0$: $[3,2,8] = [0, 0, 3] + \vec{C}$

$$[3,2,5] = \vec{C}$$

$$\vec{r}(t) = [t^3, t^2, 3e^{2t}] + [3,2,5]$$

or $\vec{r}(t) = [t^3+3, t^2+2, 3e^{2t}+5]$

Ex: Find $\int_1^2 [4t, 7]\,dt$

$$= [2t^2, 7t]\Big|_1^2$$

$$= [8, 14] - [2, 7]$$

$$= [6, 7]$$

12.3 Velocity and Acceleration

Consider an object moving in 2D or 3D.

$\vec{r}(t)$ is the position vector
$\vec{v}(t)$ " velocity "
$\vec{a}(t)$ " acceleration "
$\|\vec{v}(t)\|$ is the speed (<u>not</u> a vector)

FACTS

$$\vec{v}(t) = \vec{r}\,'(t)$$
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$
$$\vec{v}(t) = \int \vec{a}(t)\,dt$$
$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

<u>Ex</u>: Given $\vec{r} = [t^3, t^2, t]$.
Find the speed at $t=1$.
Units are m and s.

$$\vec{v} = [3t^2, 2t, 1]$$
$$\vec{v}(1) = [3, 2, 1]$$
$$\|\vec{v}(1)\| = \sqrt{14} \text{ m/s}$$

<u>Ex</u>: Given $\vec{a} = [2, 4\cos 2t, 9e^{3t}]$,
$\vec{v}(0) = [1, -1, 0]$,
$\vec{r}(0) = [2, 1, 3]$.
Find $\vec{r}(t)$.

$$\vec{v} = [2t, 2\sin 2t, 3e^{3t}] + \vec{c}$$

Sub $t=0$: $[1, -1, 0] = [0, 0, 3] + \vec{c}$

$$[1, -1, -3] = \vec{c}$$

$$\vec{v} = [2t+1, 2\sin 2t - 1, 3e^{3t} - 3]$$

$$\vec{r} = [t^2 + t, -\cos 2t - t, e^{3t} - 3t] + \vec{c}$$

Sub $t=0$: $[2, 1, 3] = [0, -1, 1] + \vec{c}$

$$[2, 2, 2] = \vec{c}$$

$$\vec{r} = [t^2 + t + 2, -\cos 2t - t + 2, e^{3t} - 3t + 2]$$

ASIDE

Position at $t=1$

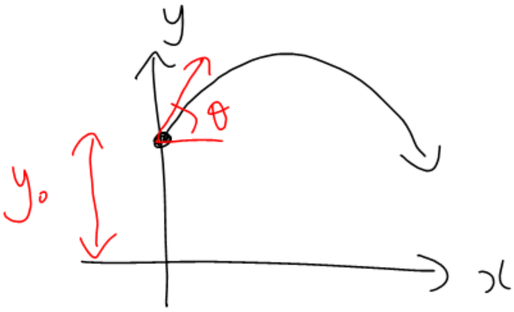
$$\vec{r}(1) = [4, -\cos 2 - 1 + 2, e^3 - 3 + 2]$$

Speed at $t=1$

$$\vec{v}(1) = [3, 2\sin 2 - 1, 3e^3 - 3]$$

$$\|\vec{v}(1)\| = \sqrt{9 + (2\sin 2 - 1)^2 + (3e^3 - 3)^2}$$

Projectile Motion in 2D



Let v_0 = initial speed

θ = angle of Inclination (measured to horizontal)

y_0 = initial height

$g = 9.8 \text{ m/s}^2$ or 32 ft/s^2

Position (in m) after t seconds:

$$\vec{r} = \left[(v_0 \cos\theta) t \; , \; y_0 + (v_0 \sin\theta) t - \frac{gt^2}{2} \right]$$

why $-\frac{gt^2}{2}$?

$$\vec{a} = [0, \; -g]$$

$$\vec{v} = [0, -gt] + \vec{C}$$

$$\vec{r} = \left[0, -\frac{gt^2}{2}\right] + \vec{C} t + \vec{C}_1$$

Ex: A projectile has

$v_0 = 10 \text{ m/s}$, $\theta = 30^\circ$, $y_0 = 2\text{m}$.

a) Find $\vec{r}$

$$\vec{r} = [(v_0 \cos\theta)t,\ y_0 + (v_0 \sin\theta)t - \tfrac{gt^2}{2}]$$

$$= [(10 \cos 30^\circ)t,\ 2 + (10 \sin 30^\circ)t - 4.9t^2]$$

$$= [5\sqrt{3}\, t,\ 2 + 5t - 4.9t^2]$$

b) Find $\vec{v}$

$$\vec{v} = [5\sqrt{3},\ 5 - 9.8t]$$

c) Find the maximum height.

Occurs when (y-component of $\vec{v}$) = 0

$$5 - 9.8t = 0$$

$$5 = 9.8t$$

$$t = \frac{5}{9.8}$$

$$\text{Maximum height} = (\text{y-component of } \vec{r})\Big|_{t=\frac{5}{9.8}}$$

$$= 2 + 5t - 4.9t^2 \Big|_{t=\frac{5}{9.8}}$$

$$\approx 3.3 \text{ m}$$

d) Find the speed when it hits the ground.

Occurs when (y-component of $\vec{r}$) = 0

$$2 + 5t - 4.9t^2 = 0$$

$$-4.9t^2 + 5t + 2 = 0$$

$$t = \frac{-5 \pm \sqrt{5^2 - 4(-4.9)(2)}}{-9.8}$$

$$\approx 1.328, \cancel{-0.307}$$

$$\bar{v}(t) = [5\sqrt{3},\ 5 - 9.8t]$$

$$\|\bar{v}(1.328)\| = \sqrt{(5\sqrt{3})^2 + (5 - 9.8 \times 1.328)^2}$$

$$\approx 12 \text{ m/s}$$

ASIDE