Test 4

Mon Dec 4

9.9 – 9.10, 10.2 – 10.5, 12.1 – 12.2

6 Questions

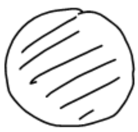
Bring: calculator
music/earplugs

Practice Problems on website

10.5 Cont'd

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

Why?

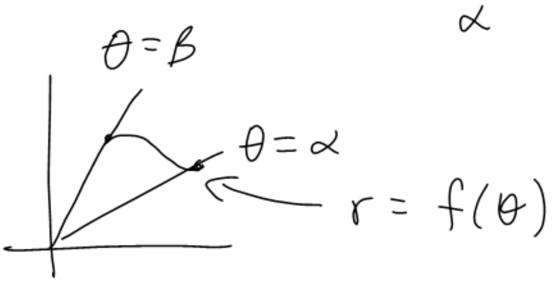


$$A_{circle} = \pi r^2$$

$$A_{sector} = \pi r^2 \left(\frac{\theta}{2\pi}\right)$$
$$= \frac{1}{2} r^2 \theta$$

$$dA = \frac{1}{2} r^2 \, d\theta$$

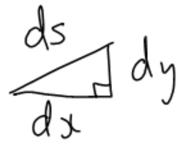
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$
$$= \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$



Arc Length $\quad s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Why?

$$ds = \sqrt{dx^2 + dy^2}$$



$$= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = r\cos\theta \qquad y = r\sin\theta$

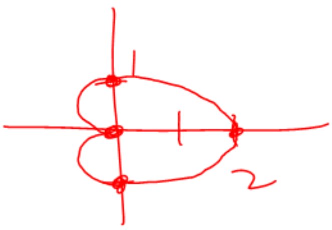
$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta \frac{dr}{d\theta} \qquad \frac{dy}{d\theta} = \dots$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

<u>Ex</u>: Find the arc length of $r = 1 + \cos\theta$



$r = 1 + \cos\theta$

Curve is traced out over $0 \le \theta \le 2\pi$.

By symmetry, $s =$ double the arc length over $0 \le \theta \le \pi$.

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$\frac{dr}{d\theta} = -\sin\theta$$

$$S = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta} \, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta} \, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2+2\cos\theta} \, d\theta$$

$$\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

$$\frac{1}{2} + \frac{\cos 2\theta}{2} = \cos^2\theta$$

$$\frac{1}{2} + \frac{\cos 2\alpha}{2} = \cos^2\alpha$$

$$\alpha = \frac{\theta}{2}: \quad \frac{1}{2} + \frac{\cos\theta}{2} = \cos^2\frac{\theta}{2}$$

$$2 + 2\cos\theta = 4\cos^2\frac{\theta}{2}$$

$$= 2\int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}} \, d\theta$$

$$= 2\int_0^{\pi} \left|2\cos\frac{\theta}{2}\right| d\theta$$

$$= 2 \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$= 8(1-0)$$

$$= 8$$

12.1 Vector-Valued Functions

Recall: parametric curve (Section 10.2)

$$\begin{cases} x = x(t) \\ y = y(t) \\ a \le t \le b \end{cases}$$

Can be described with the position vector

$$\vec{r}(t) = [x(t), y(t)] \qquad (a \le t \le b)$$

or $$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} \qquad (a \le t \le b)$$

Ex: $\vec{r}(t) = [t, t^2]$

Find $\vec{r}(6)$

$\vec{r}(6) = [6, 36]$ This is the position at $t=6$.

Note: $\vec{r}(t)$ is called a vector-valued function.

Ex: Find the position vector:

$$\begin{cases} x = 4+2t \\ y = 1+5t \\ z = 3+t \\ 0 \le t \le 1 \end{cases}$$

$$\vec{r}(t) = [4+2t, 1+5t, 3+t] \quad (0 \le t \le 1)$$

or $$\vec{r}(t) = (4+2t)\vec{i} + (1+5t)\vec{j} + (3+t)\vec{k} \quad (0 \le t \le 1)$$

Ex: Find the position vector of the line segment from $(1,2,3)$ to $(-4,6,8)$.

x: start at 1, net change of -5
y: 2, $+4$
z: 3, $+5$

$$\begin{cases} x = 1-5t \\ y = 2+4t \\ z = 3+5t \\ 0 \le t \le 1 \end{cases}$$

$$\vec{r}(t) = [1-5t, 2+4t, 3+5t] \quad (0 \le t \le 1)$$

12.2 Derivatives and Integrals of Vector-Valued Functions

FACT

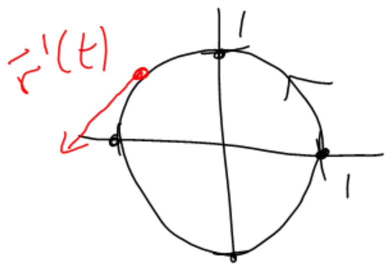
If $\vec{r}(t) = [x(t), y(t), z(t)]$

then $\vec{r}'(t) = [x'(t), y'(t), z'(t)]$

<u>Note</u>: $\vec{r}'(t)$ is a tangent vector to the curve.

<u>Ex</u>: Find $\vec{r}'(t)$ for $\vec{r}(t) = [\cos t, \sin t]$

$\vec{r}'(t) = [-\sin t, \cos t]$



6 Properties

Let : c be a constant

$f(t)$ " function of t

$\vec{r}(t), \vec{s}(t)$ be vector-valued functions

1) $[c\,\vec{r}(t)]' = c\,\vec{r}'(t)$

2) $[\vec{r}(t) \pm \vec{s}(t)]' = \vec{r}'(t) \pm \vec{s}'(t)$

3) $[f(t)\,\vec{r}(t)]' = f(t)\,\vec{r}'(t) + f'(t)\,\vec{r}(t)$

Product Rule

4) $[\vec{r}(t) \cdot \vec{s}(t)]' = \vec{r}(t) \cdot \vec{s}'(t) + \vec{r}'(t) \cdot \vec{s}(t)$

5) $[\vec{r}(t) \times \vec{s}(t)]' = \vec{r}(t) \times \vec{s}\,'(t) + \vec{r}\,'(t) \times \vec{s}(t)$

6) $\frac{d}{dt} \vec{r}(f(t)) = \vec{r}\,'(f(t))\, f'(t)$

Chain Rule

Ex: Find $\frac{d}{dt} t^2 [9t, t^3]$ two different ways.

$$\frac{d}{dt} t^2 [9t, t^3] = \frac{d}{dt} [9t^3, t^5]$$
$$= [27t^2, 5t^4]$$

$$\frac{d}{dt} t^2 [9t, t^3] = t^2 [9, 3t^2] + 2t [9t, t^3]$$
$$= [9t^2, 3t^4] + [18t^2, 2t^4]$$
$$= [27t^2, 5t^4]$$

Ex: Let $\vec{r}(t) = [t^2+1, 7t]$

Find $\frac{d}{dt} \vec{r}(2t)$ two different ways.

$$\frac{d}{dt} \vec{r}(2t) = \frac{d}{dt} [4t^2+1, 14t]$$
$$= [8t, 14]$$

$$\frac{d}{dt} \vec{r}(2t) = \vec{r}\,'(2t)\,(2)$$

$$\boxed{\vec{r}\,'(t) = [2t, 7]}$$

$$= [4t, 7]\,(2)$$

$$= [8t, 14]$$