10.5 Cont'd

Ex: Find the area of one petal of

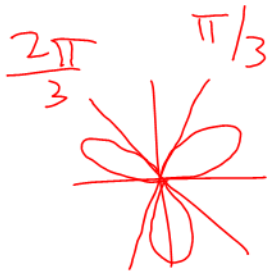
$r = \sin 3\theta$

Solve $r=0$ to find where the petals start.

$0 = \sin 3\theta$

$3\theta = 0, \pi, 2\pi, \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$



Integrate over $0 \le \theta \le \frac{\pi}{3}$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

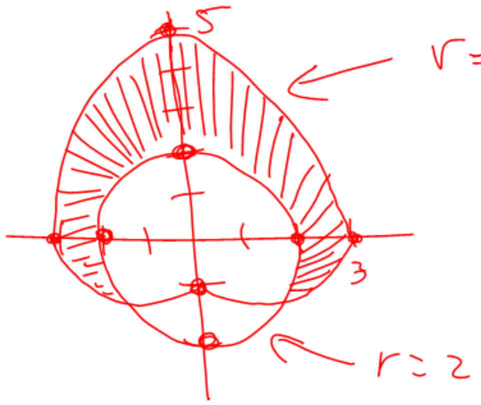
$$= \frac{1}{2}\int_0^{\pi/3} \left[\frac{1}{2} - \frac{\cos 6\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\left[\frac{\theta}{2} - \frac{\sin 6\theta}{12}\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0\right]$$

$$= \frac{\pi}{12}$$

Ex: Find the area inside $r = 3 + 2\sin\theta$ and outside $r = 2$.



Intersection : $r=r$

$$3+2\sin\theta = 2$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6},\ -\frac{\pi}{6}$$
or $$\theta = \frac{7\pi}{6},\ \frac{11\pi}{6}$$

Caution: θ must be increasing
$$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$$

$\boxed{\text{area inside } r=3+2\sin\theta}$ $\boxed{\text{area inside } r=2}$

$$A = \underbrace{\frac{1}{2}\int_{-\frac{\pi}{6}}^{7\pi/6} (3+2\sin\theta)^2\,d\theta}_{A_1} - \underbrace{\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} 2^2\,d\theta}_{A_2}$$

$$A_1 = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left(3 + 2\sin\theta\right)^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left(9 + 12\sin\theta + \underbrace{4\sin^2\theta}_{2-2\cos 2\theta}\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left(11 + 12\sin\theta - 2\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\Big[11\theta - 12\cos\theta - \sin 2\theta\Big]_{-\pi/6}^{7\pi/6}$$

$$= \frac{1}{2}\left[\frac{77\pi}{6} - 12\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} - \left(-\frac{11\pi}{6} - 12\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{88\pi}{6} + 22\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{22\pi}{3} + \frac{11\sqrt{3}}{2}$$

$$A_2 = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 4\, d\theta$$

$$= \frac{1}{2}\left[4\theta\right]_{-\pi/6}^{7\pi/6}$$

$$= 2\left[\frac{7\pi}{6} + \frac{\pi}{6}\right]$$

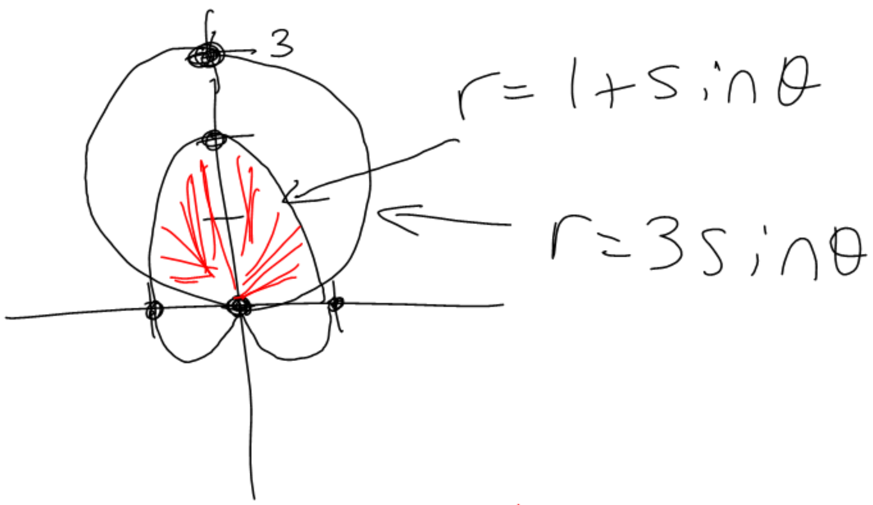
$$= \frac{16\pi}{6}$$

$$= \frac{8\pi}{3}$$

$$A = A_1 - A_2$$

$$= \frac{14\pi}{3} + \frac{11\sqrt{3}}{2}$$

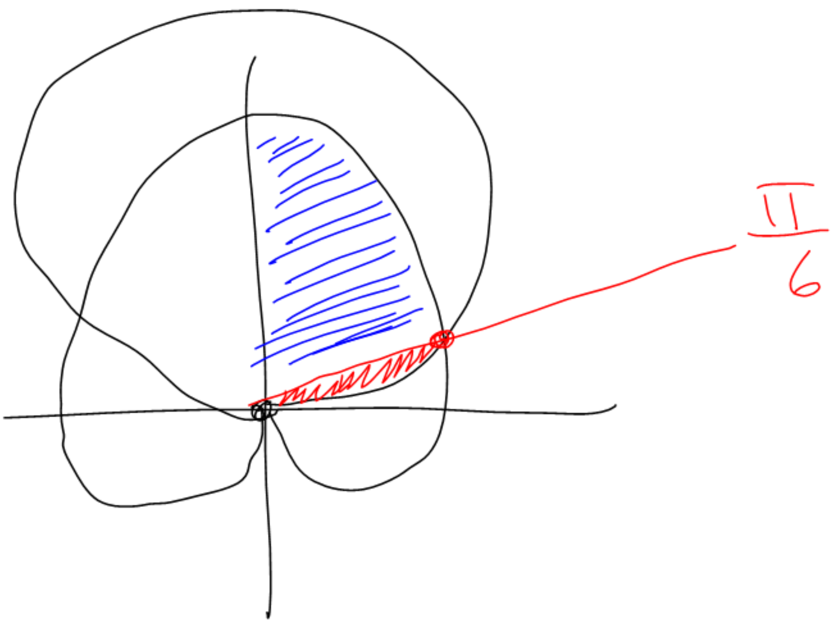
Ex: Find the area inside both $r = 1+\sin\theta$ and $r = 3\sin\theta$.



By symmetry, the area is double the area in $0 \le \theta \le \frac{\pi}{2}$

Intersection $r = r$

$$1 + \sin\theta = 3\sin\theta$$
$$1 = 2\sin\theta$$
$$\frac{1}{2} = \sin\theta$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$



Over $0 \le \theta \le \frac{\pi}{6}$, $r = 3\sin\theta$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$, $r = 1 + \sin\theta$

$$A = \underbrace{\int_0^{\pi/6} (3\sin\theta)^2 \, d\theta}_{A_1} + \underbrace{\int_{\pi/6}^{\pi/2} (1+\sin\theta)^2 \, d\theta}_{A_2}$$

$$A_1 = \int_0^{\pi/6} 9\sin^2\theta \, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6} (1-\cos 2\theta)\, d\theta$$

$$\vdots$$

$$= \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}$$

$$A_2 = \int_{\pi/6}^{\pi/2} \left(1 + 2\sin\theta + \underbrace{\sin^2\theta}_{\frac{1}{2} - \frac{\cos 2\theta}{2}}\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \left[\frac{3}{2}\theta - 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\pi/6}^{\pi/2}$$

$$= \left[\frac{3\pi}{4} - \left(\frac{3\pi}{12} - 2\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{3\pi}{4} - \frac{\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{8}$$

$$= \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$$

$$A = A_1 + A_2$$

$$= \frac{3\pi}{4} + \frac{\pi}{2}$$

$$= \frac{5\pi}{4}$$