10.4 Cont'd

FACT

$r = f(\theta)$ as a parametric curve:

$$\begin{cases} x = r\cos\theta \\ \quad = f(\theta)\cos\theta \\ y = r\sin\theta \\ \quad = f(\theta)\sin\theta \end{cases}$$

It has $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

Recall: Vertical tangent if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

Horizontal " $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

Ex: Given $r = 1 + \cos\theta$

For which θ is the tangent vertical or horizontal?

$x = r\cos\theta$

$= (1+\cos\theta)\cos\theta$

$= \cos\theta + \cos^2\theta$

$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta(-\sin\theta)$

$\frac{dx}{d\theta} = 0$: $0 = -\sin\theta(1 + 2\cos\theta)$

$\sin\theta = 0$ or $1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

reference angle $= \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

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$$y = r\sin\theta$$
$$= (1+\cos\theta)\sin\theta$$

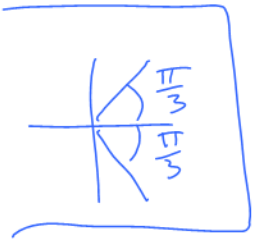
$$\frac{dy}{d\theta} = (1+\cos\theta)\cos\theta + \sin\theta(-\sin\theta) \quad \text{Product Rule}$$
$$= \cos\theta + \cos^2\theta \underbrace{-\sin^2\theta}_{-(1-\cos^2\theta)}$$

$$= -1 + \cos\theta + 2\cos^2\theta$$

$\frac{dy}{d\theta} = 0$:

$$0 = 2\cos^2\theta + \cos\theta - 1$$
$$0 = (2\cos\theta - 1)(\cos\theta + 1)$$

$$2\cos\theta - 1 = 0 \qquad \cos\theta + 1 = 0$$
$$\cos\theta = \frac{1}{2} \qquad \cos\theta = -1$$



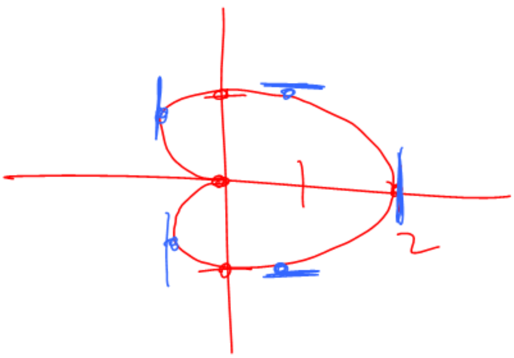
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$$

Vertical Tangent $\Rightarrow$ $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Horizontal Tangent $\Rightarrow$ $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

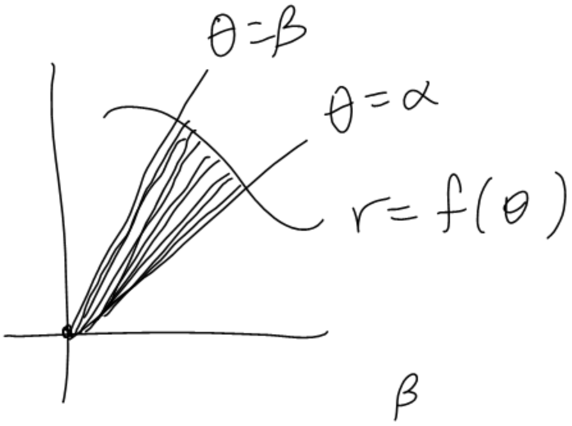
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$



$$r = 1 + \cos\theta$$

θ	r
0	2
$\pi/2$	1
π	0
$3\pi/2$	1

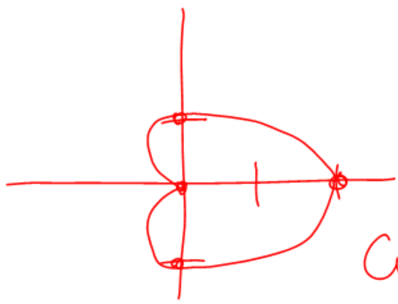
10.5 Area and Arc Length in Polar



Area $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$

Ex: Find the area inside $r = 1+\cos\theta$.

θ	r
0	2
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	1



Curve is traced out over $0 \le \theta < 2\pi$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} [1+\cos\theta]^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} (1+2\cos\theta+\underbrace{\cos^2\theta}_{\frac{1}{2}+\frac{\cos 2\theta}{2}}) \, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} \left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right) d\theta$$

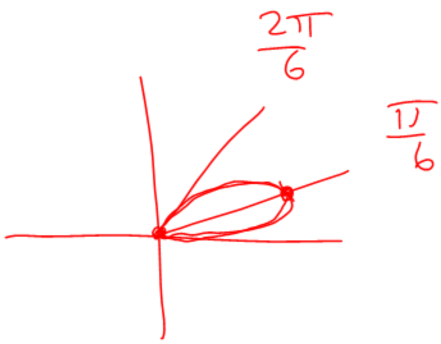
$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left[3\pi - 0\right]$$

$$= \frac{3\pi}{2}$$

Ex: Find the area of one petal of $r = \sin 3\theta$.

θ	r
0	0
$\frac{\pi}{6}$	1
$\frac{2\pi}{6}$	0
$\frac{3\pi}{6}$	-1



OR

Solve $r=0$ to find where petals start.

$\sin 3\theta = 0$

$3\theta = 0, \pi, 2\pi, 3\pi, \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$

$$0 \le \theta \le \frac{\pi}{3}$$

$$A = \frac{1}{2}\int_0^{\pi/3} \left[\sin 3\theta\right]^2 d\theta$$

To Be Continued