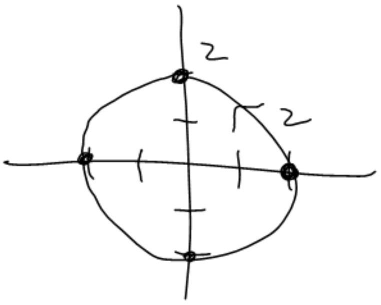
10.3 Parametric Curves and Calculus

Parametric curve:

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ 0 \le t < 2\pi \end{cases}$$



FACT

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Comes from the Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Ex: Find the slope of the tangent line at $t=1$:

$$\begin{cases} x = 2t^2+1 \\ y = t^3 + t^5 \\ -\infty < t < \infty \end{cases}$$

$$\frac{dy}{dt} = 3t^2 + 5t^4 \qquad \frac{dx}{dt} = 4t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{3t^2+5t^4}{4t}$$

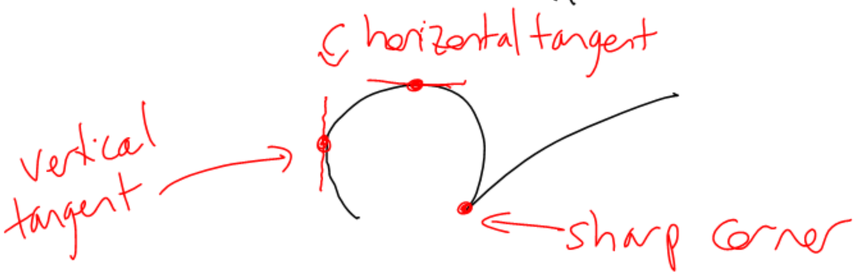
$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8}{4}$$
$$= 2$$

FACT

If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ then there is a horizontal tangent.

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ " vertical "

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ then there <u>may</u> be a sharp corner.



Why?

$$\text{Horizontal tangent} \iff \frac{dy}{dx} = 0$$
$$\iff \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = 0$$
$$\iff \frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0$$

<u>Ex:</u>
$$\begin{cases} x = 4t + t^4 \\ y = 1 + t^2 \\ -\infty < t < \infty \end{cases}$$

Find all points (x, y) where there is a horizontal or vertical tangent.

$$\frac{dy}{dt} = 2t$$

Set $\frac{dy}{dt} = 0$:

$$2t = 0$$
$$t = 0$$

$$\frac{dx}{dt} = 4 + 4t^3$$

Set $\frac{dx}{dt} = 0$:

$$4 + 4t^3 = 0$$
$$4t^3 = -4$$
$$t^3 = -1$$
$$t = -1$$

Horizontal Tangent

$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\Rightarrow t = 0$

$\Rightarrow (x, y) = (0, 1)$

Vertical Tangent

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\Rightarrow t = -1$

$\Rightarrow (x, y) = (-3, 2)$

FACT

The second derivative $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$

Recall: The curve is <u>Concave up</u> if $\dfrac{d^2y}{dx^2} > 0$ $\cup$

" <u>Concave down</u> if $\dfrac{d^2y}{dx^2} < 0$ $\cap$

<u>Ex</u>: $\begin{cases} x = t^2+1 \\ y = t^6+5 \\ -\infty < t < \infty \end{cases}$

Find $\dfrac{d^2y}{dx^2}$

$\dfrac{dy}{dt} = 6t^5$ $\qquad$ $\dfrac{dx}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{6t^5}{2t}$$

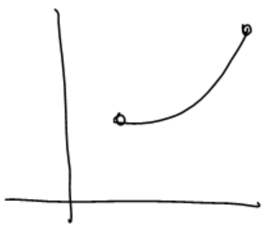
$$= 3t^4$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{12t^3}{2t}$$

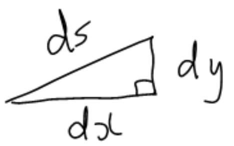
$$= 6t^2$$

Consider $\begin{cases} x = x(t) \\ y = y(t) \\ a \le t \le b \end{cases}$



Arc Length $\quad s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

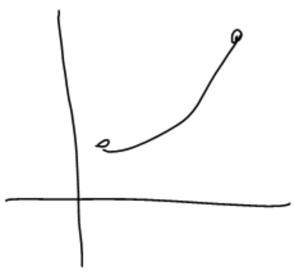
Why?



$$ds = \sqrt{dx^2 + dy^2}$$

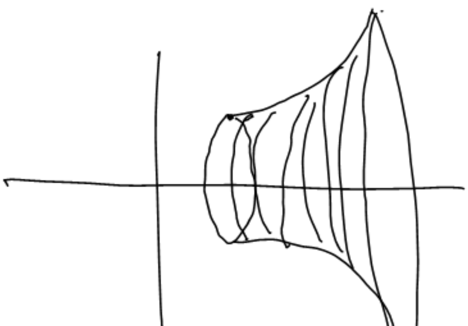
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



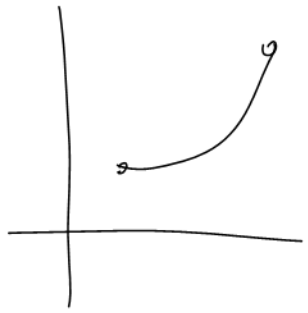
Parametric Curve

Revolve about x-axis:



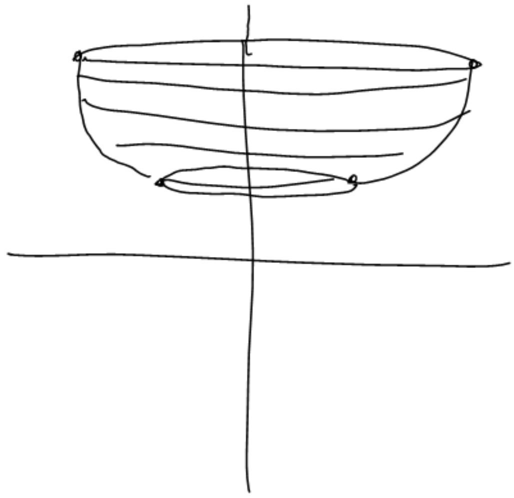
SURFACE (2D)

Surface Area $\quad S_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$



Parametric Curve

Revolve about y-axis:



SURFACE (2D)

Surface Area $S_y = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

Ex: Find the arc length of:

$$\begin{cases} x = t^2 \\ y = t^3 \\ 1 \le t \le 2 \end{cases}$$

$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 3t^2$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 \sqrt{t^2(4+9t^2)}\, dt$$

$$= \int_1^2 \sqrt{t^2}\sqrt{4+9t^2}\, dt$$

$|t|$
$= t$ if $t > 0$

$$= \int_1^2 t\sqrt{4+9t^2}\, dt$$

$u = 4+9t^2$
$du = 18t\,dt$
$\frac{du}{18} = t\,dt$

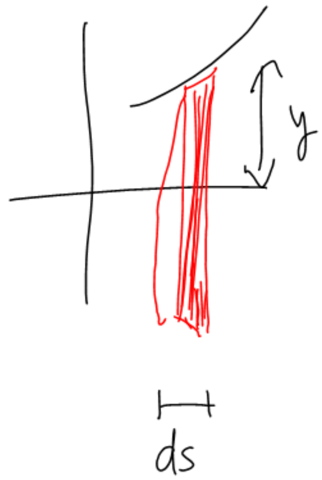
$t=1 \Rightarrow u=13$
$t=2 \Rightarrow u=40$

$$= \frac{1}{18}\int_{13}^{40} u^{1/2}\, du$$

$$= \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{13}^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right)$$ ✓

ASIDE Where do surface area formulas come from?



$$dS_x = \text{area of ring}$$

$$= 2\pi y \, ds$$

$$S_x = 2\pi \int y \, ds$$

$$= 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$