1. Decide whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{7\sqrt{n}+8}{8n+7\sqrt{n}}$$

Limit Comparison Test

Dominant terms $= \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

Compare it with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\frac{7\sqrt{n}+8}{8n+7\sqrt{n}} > 0$ and $\frac{1}{\sqrt{n}} > 0$ for $n \geq 1$ ✓

$$L = \lim_{n\to\infty} \frac{\left(\frac{7\sqrt{n}+8}{8n+7\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)}$$

$$= \lim_{n\to\infty} \left(\frac{7\sqrt{n}+8}{8n+7\sqrt{n}}\right)\left(\frac{\sqrt{n}}{1}\right)$$

$$= \lim_{n\to\infty} \frac{7n+8\sqrt{n}}{8n+7\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{7+\frac{8}{\sqrt{n}}}{8+\frac{7}{\sqrt{n}}}$$

$$= \frac{7}{8}$$

$0 < L < \infty$ ✓

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series)

$$\Rightarrow \sum_{n=1}^{\infty} \frac{7\sqrt{n}+8}{8n+7\sqrt{n}} \text{ diverges}$$

(2) a) Find the 3rd degree Taylor polynomial for $f(x) = \ln x$ centred at $c = 1$.

$f(x) = \ln x$ $\qquad f(1) = 0$

$f'(x) = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = -x^{-2}$ $\qquad f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\qquad f'''(1) = 2$

$$P_3(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$

$$= 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 \checkmark$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \checkmark$$

b) Find an upper bound for the error $|R_3(0.8)|$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$|R_3(x)| = \left| \frac{f^{(4)}(z)}{4!}(x-c)^4 \right|$$

$$f^{(4)}(x) = -6x^{-4}$$

$$f^{(4)}(z) = -6z^{-4}$$

$$|R_3(x)| = \left| \frac{6z^{-4}}{24}(x-c)^4 \right|$$

$$|R_3(0.8)| = \left| \frac{6z^{-4}}{24}(0.2^4) \right|$$

where z is between $\cancel{x}$ 0.8 and $\cancel{c}$ 1

$$\le \frac{6(0.8)^{-4}}{24}(0.2)^4$$

$$\le 0.000977$$

ASIDE

$$\left| \underbrace{\ln 0.8}_{\text{true}} - \left[\underbrace{(0.8-1) - \tfrac{1}{2}(0.8-1)^2 + \tfrac{1}{3}(0.8-1)^3}_{\text{approximation}} \right] \right| \le \underbrace{0.000977}_{\text{error}}$$

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(3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ Converges by the Alternating Series Test.

Find the smallest N so that

$$|R_N| \le 0.01$$

For an alternating series $a_n = |n^{th} \text{ term}|$

$$|R_N| \le a_{N+1}$$

Plan: $a_{N+1} \le 0.01$

$$\frac{1}{(N+1)^2+1} \le 0.01$$

$$\boxed{a_n = \frac{1}{n^2+1} \qquad a_{N+1} = \frac{1}{(N+1)^2+1}}$$

$$\frac{1}{\cancel{0.01}\ 100} \leq (N+1)^2 + 1$$

$$99 \leq (N+1)^2$$

$$\sqrt{99} \leq N+1$$

$$\sqrt{99} - 1 \leq N$$

$$N \geq 8.95$$

$$\boxed{N = 9}$$

4. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges by the Integral Test.

a) Find an upper bound for R_5.

$$R_5 \leq \int_5^{\infty} \frac{1}{x^3}\,dx$$

$$\leq \lim_{b\to\infty} \int_5^b x^{-3}\,dx$$

$$\leq \lim_{b\to\infty} -\frac{1}{2}x^{-2} \Big|_5^b$$

$$\leq \lim_{b\to\infty} -\frac{1}{2}b^{-2} + \frac{1}{2}5^{-2}$$

$$\leq 0.02$$

b) Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$, given $S_5 \approx 1.1857$

$$S_5 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq S_5 + R_5$$

⑤ a) For which series can we calculate the sum?

Geometric and Telescoping

b) Find $\sum_{n=2}^{\infty} \frac{13}{7}(0.89)^{n-1}$

$= \frac{a}{1-r}$

$= \frac{\frac{13}{7}(0.89)}{0.11}$

≈ 15.03

c) Find $\sum_{n=7}^{\infty} \frac{8}{(n+4)(n+5)}$

$\vdots$

$= \sum_{n=7}^{\infty} \left[\frac{8}{n+4} - \frac{8}{n+5}\right]$

$= \frac{8}{11} \cancel{- \frac{8}{12} + \frac{8}{12}} - \frac{8}{13} + \ldots$

$= \frac{8}{11} - \lim_{n\to\infty} \frac{8}{n+5}$

$= \frac{8}{11}$

ASIDE

$\sum_{n=1}^{\infty} \frac{13}{7}(0.89)^{n-1}$

$a = \frac{13}{7}$ $r = 0.89$

$\sum_{n=0}^{\infty} \frac{13}{7}(0.89)^{n-1}$

$a = \frac{13}{7}(0.89)^{-1}$

$r = 0.89$

$\sum_{n=6}^{\infty} (0.62)^{2n+1}$

$= 0.62^{13} + 0.62^{15} + \ldots$

$a = 0.62^{13}$

$r = 0.62^{2}$

⑥ Find the sum $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

$\frac{4}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$

$$4 = A(n+2) + Bn$$

$n=0$: $$4 = 2A \Rightarrow A = 2$$

$n=-2$: $$4 = -2B \Rightarrow B = -2$$

$$\text{Series} = \sum_{n=1}^{\infty} \left[\frac{2}{n} - \frac{2}{n+2} \right]$$

$$= \underbrace{\frac{2}{1} - \cancel{\frac{2}{3}} + \frac{2}{2}} - \cancel{\frac{2}{4}} + \cancel{\frac{2}{3}} - \frac{2}{5} + \cancel{\frac{2}{4}} - \frac{2}{6} + \dots$$

$$= 3 - \lim_{n \to \infty} \left(\frac{2}{n} + \frac{2}{n+1} \right)$$

$$= 3$$