9.10 Taylor and Maclaurin Series

The Taylor series of f centred at c:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ then it's called the Maclaurin series of f.

Ex: Find the Maclaurin series of $\sin x$.

$f(x) = \sin x$ $\qquad$ $f(0) = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad$ $f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\qquad$ $f^{(4)}(0) = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0 + \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \checkmark$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \checkmark$$

Comments:

- Interval of Convergence for above series is $-\infty < x < \infty$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ (not an approximation)

- If a question asks you to find a series, you can give the first 3 nonzero terms.

Ex: Find the Maclaurin series of $(1+x)^k$, where k is a real number.

$f(x) = (1+x)^k \qquad f(0) = 1$

$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0) = k(k-1)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + \frac{k}{1!}x + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$$

Ex: Find the Maclaurin series for $x^3 \sin x^2$, given $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$x^3 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}$$

Ex: Find the first three nonzero terms of the Maclaurin series for $e^x\sqrt{1+x}$. Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$

$$e^x = 1 + x + \frac{x^2}{2} + \ldots$$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$$

$$e^x(1+x)^{1/2} = \left(1 + x + \frac{x^2}{2} + \ldots\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + \ldots\right)$$

$$\begin{array}{lllll} = & 1 & + & \frac{x}{2} & -\frac{x^2}{8} \\ & & & x & \frac{x^2}{2} \\ & & & & \frac{x^2}{2} \end{array}$$

$$= 1 + \frac{3x}{2} + \frac{7x^2}{8} + \ldots$$

Ex: a) Approximate $\int_0^{0.5} e^{-x^2}\,dx$ using 4 nonzero terms.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{6} + \dots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

$$\int_0^{0.5} e^{-x^2} dx \approx \int_0^{0.5} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx$$

$$\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42}$$

$$\approx 0.46127232$$

b) Find an upper bound for |error|

$0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} + \dots$ is alternating

$$|R_N| \leq a_{N+1}$$

|error| $\leq$ absolute value of next term

next term of e^x : $\frac{x^4}{24}$

e^{-x^2} : $\frac{x^8}{24}$

$$\int e^{-x^2}dx : \frac{x^9}{9(24)}$$

$$\int_0^{0.5} e^{-x^2}dx : \frac{0.5^9}{9(24)}$$

$$|\text{error}| \le \frac{0.5^9}{9(24)}$$

$$\le 9.0 \times 10^{-6}$$

c) Conclusion about $\int_0^{0.5} e^{-x^2}dx$?

$$\int_0^{0.5} e^{-x^2}dx = 0.46127232 \pm 9.0 \times 10^{-6}$$

d) How many nonzero terms are required for $|\text{error}| \le 0.01$?

$$|R_N| \le a_{N+1}$$

$$\int_0^{0.5} e^{-x^2}dx \approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \ldots$$

≤ 0.01
(in absolute value)

$\boxed{2 \text{ terms}}$

9.10 #47

Find the first 4 nonzero terms of $e^x \sin x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)$$

$$= \begin{array}{ccccc} x & & -\frac{x^3}{6} & & +\frac{x^5}{120} \\ & x^2 & & -\frac{x^4}{6} & \\ & & \frac{x^3}{2} & & -\frac{x^5}{12} \\ & & & \frac{x^4}{6} & \\ & & & & \frac{x^5}{24} \end{array}$$

$$= x + x^2 + \frac{x^3}{3} \qquad - \frac{x^5}{30} + \dots$$