9.9 Cont'd

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Valid: $-1 < x < 1$

Ex: Find a power series for $\arctan x$ centred at $c = 0$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\arctan x + C_1 = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n \,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Sub $x = 0$: $\quad 0 + C_1 = 0$

$$C_1 = 0$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Ex: a) Approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^5}\,dx$

using 3 terms of a power series.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - \dots$$

$$\frac{1}{1+x^5} = 1 - x^5 + x^{10} - \dots$$

$$\int_0^{0.5} \frac{1}{1+x^5}\,dx \approx \int_0^{0.5} (1 - x^5 + x^{10})\,dx$$

$$\approx \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} \right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^6}{6} + \frac{0.5^{11}}{11}$$

$$\approx 0.49744022$$

b) Find an upper bound for |error|

$0.5 - \frac{0.5^6}{6} + \frac{0.5^{11}}{11} - \dots$ is an alternating series.

$$|R_N| \le a_{N+1}$$

absolute value of next term

$$|\text{error}| \le \frac{0.5^{16}}{16}$$

$$\le 9.6 \times 10^{-7}$$

c) Conclusion about the integral?

$$0.49744022 - 9.6\times10^{-7} \le \int_0^{0.5} \frac{1}{1+x^5}\,dx \le 0.49744022 + 9.6\times10^{-7}$$

FUN FACT ABOUT π (Won't Be Tested)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$x=1$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$$

9.9 # 11

Find a power series for $f(x) = \frac{5}{2x-3}$ centred at $c=-3$ and find the interval of convergence.

$$f(x) = \frac{5}{2(x+3) + ?}$$

$$= \frac{5}{2(x+3) - 9}$$

$$= \frac{5}{9}\left[\frac{1}{\frac{2}{9}(x+3) - 1}\right]$$

$$= \frac{-5}{9}\left[\frac{1}{1 - \frac{2}{9}(x+3)}\right]$$

$$= \frac{-5}{9}\sum_{n=0}^{\infty}\left[\frac{2}{9}(x+3)\right]^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left[\frac{2}{9}(x+3)\right]^{n+1}}{\left[\frac{2}{9}(x+3)\right]^n}\right|$$

$$= \left| \frac{2}{9}(x+3) \right|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2}{9}(x+3) \right|$$

Series converges if $\left| \frac{2}{9}(x+3) \right| < 1$

$$|x+3| < \frac{9}{2}$$

$$-\frac{9}{2} < x+3 < \frac{9}{2}$$

$$-\frac{15}{2} < x < \frac{3}{2}$$

$x = -\frac{15}{2}$:

$$\text{Series} = \sum_{n=0}^{\infty} \left[\frac{2}{9}\left(-\frac{9}{2}\right)\right]^n$$

$$= \sum_{n=0}^{\infty} [-1]^n$$

Diverges (nth term test)

$x = \frac{3}{2}$:

$$\text{Series} = \sum_{n=0}^{\infty} \left[\frac{2}{9}\left(\frac{9}{2}\right)\right]^n$$

$$= \sum_{n=0}^{\infty} 1^n$$

Diverges (nth term test)

$$\boxed{-\frac{15}{2} < x < \frac{3}{2}}$$