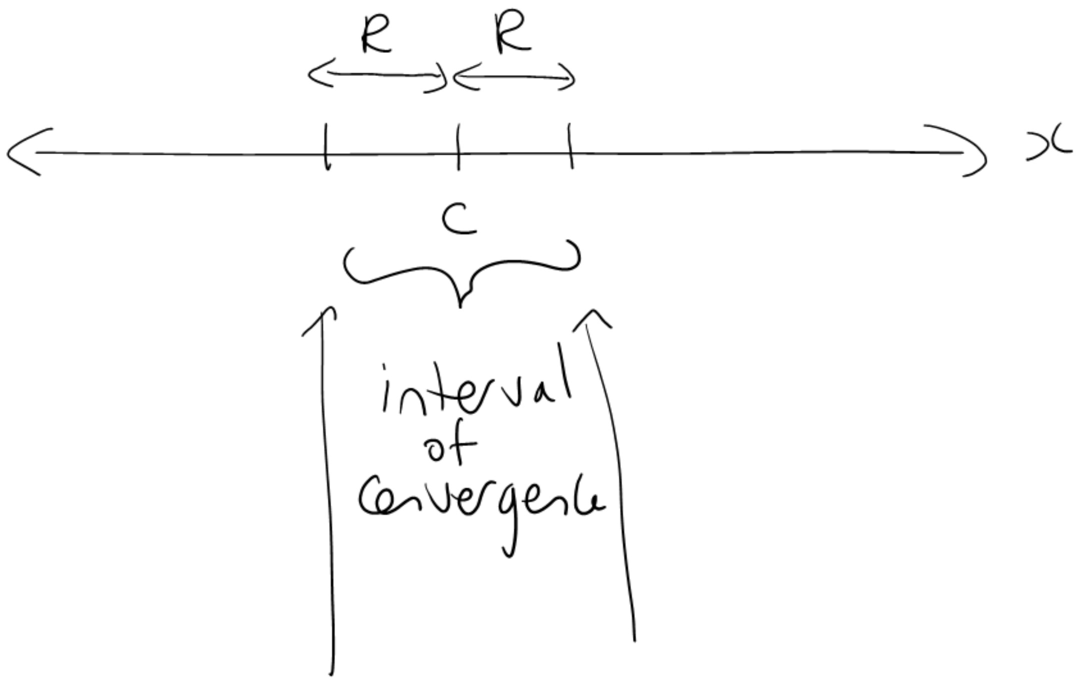
9.8 Power Series Cont'd

Let $c=$ centre of power series

$R=$ radius of convergence



Analyze the endpoints to see whether the series converges there.

Ex: Find the interval of convergence:

$$\sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \left| \frac{x}{n+1} \right|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$$

Series converges for all x

Interval of Convergence: $-\infty < x < \infty$

Radius of Convergence: ∞

Ex: Find the interval of convergence:

$$\sum_{n=0}^{\infty} \underbrace{n!\,(x-2)^n}_{a_n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n} \right|$$

$$= \left| (n+1)(x-2) \right|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} \infty & , x \neq 2 \\ 0 & , x = 2 \end{cases}$$

Interval of Convergence: $x = 2$

Radius " : $R = 0$

Ex: Find the interval of Convergence:

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{\underbrace{n \cdot 9^n}_{a_n}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+1}}{(n+1)\,9^{n+1}} \cdot \frac{n\,9^n}{(x-4)^n}\right|$$

$$= \left|\frac{(x-4)}{9} \quad \frac{n}{n+1}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-4}{9}\,(1)\right|$$

L'Hôpital's Rule

$$= \left|\frac{x-4}{9}\right|$$

Series Converges if $\left|\frac{x-4}{9}\right| < 1$

$$|x-4| < 9$$

$$-9 < x-4 < 9$$

$$-5 < x < 13$$

$x=-5$:

Series $= \sum_{n=1}^{\infty} \frac{(-9)^n}{n \cdot 9^n}$

$x=13$:

Series $= \sum_{n=1}^{\infty} \frac{\overset{1}{\cancel{9^n}}}{n \cdot \cancel{9^n}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges (Alternating)

Diverges

(p-series)

$$\boxed{-5 \le x < 13}$$

FACT

Suppose $f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$

has radius of convergence $R > 0$.

Then $f'(x)$ and $\int f(x)\,dx$ also

have radius of convergence $= R$.

On the interval $(c-R, c+R)$:

$$f'(x) = \sum_{n=0}^{\infty} n\, b_n (x-c)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{b_n (x-c)^{n+1}}{n+1} + C_1$$

Ex: Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

has $R=1$. Find $f'(x)$ and $\int f(x)\,dx$

on the interval $-1 < x < 1$:

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1}$$

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} + C_1$$

9.9 Power Series Representations

Recall $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1<r<1$

$$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } -1<x<1$$

and $\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$ for $-1<x<1$

FACT

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } -1<x<1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \quad \text{for } -1<x<1$$

Ex: Find a power series for $\frac{1}{1+x^3}$ centred at $c=0$.

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

<u>Ex</u>: Find a power series for $\frac{10}{3+2x}$ centred at $c=1$.

Centred at $c=0$:
$\sum b_n x^n$

Centred at c :
$\sum b_n (x-c)^n$

$$\frac{10}{3+2x} = \frac{10}{? + 2(x-1)}$$

$$= \frac{10}{5+2(x-1)}$$

$$= \frac{10}{5}\left[\frac{1}{1+\frac{2}{5}(x-1)}\right]$$

$$= 2\sum_{n=0}^{\infty}\left[-\frac{2}{5}(x-1)\right]^n$$

$$= 2 \sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n (x-1)^n$$

Ex: Find a power series for $\frac{1}{(1-x)^2}$ centred at $c=0$.

Notice $\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} (1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

Ex: Find a power series for $\ln(1+x)$ centred at $c=0$.

Notice: $\ln(1+x) + C_1 = \int \frac{1}{1+x} dx$

$$\int \frac{1}{1+x} dx = \ln(1+x) + C_1$$

$$\ln(1+x) + C_1 = \int \frac{1}{1+x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n \,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Sub $x=0$:

$$\underbrace{\cancel{\ln(1)}}_{0} + C_1 = 0$$

$$C_1 = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$