9.7 Taylor and Maclaurin Polynomials

The <u>n^{th} degree Taylor polynomial</u>
<u>of $f(x)$ centred at c</u> is:

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2$$
$$+ \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ then it's called the
<u>n^{th} degree Maclaurin polynomial of $f(x)$</u>.

<u>Ex</u>: Find the 7^{th} degree Maclaurin polynomial of $f(x) = \sin x$.

$f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$ $\qquad f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x$ $\qquad f^{(6)}(0) = 0$

$f^{(7)}(x) = -\cos x$ $\qquad f^{(7)}(0) = -1$

$$P_7(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(7)}(0)}{7!}x^7$$

$$= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \checkmark$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \checkmark$$

Ex: Find the n^{th} degree Taylor polynomial of $f(x) = \ln x$ centred at $c=1$

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = x^{-1}$	$f'(1) = 1$
$f''(x) = -x^{-2}$	$f''(1) = -1$
$f'''(x) = 2x^{-3}$	$f'''(1) = 2$
$f^{(4)}(x) = -6x^{-4}$	$f^{(4)}(1) = -6$
$\vdots$	$\vdots$
$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,x^{-n}$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$$P_n(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$- \frac{6}{4!}(x-1)^4 + \dots + (-1)^{n+1}\frac{(n-1)!}{n!}(x-1)^n$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$- \frac{1}{4}(x-1)^4 + \dots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder or error}}$$

FACT

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \quad \text{where } z \text{ is a number between } x \text{ and } c.}$$

Ex: a) Find the 2^{nd} degree Taylor polynomial of $f(x)=\sqrt{x}$ at $c=4$.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f(4) = 2$

$f'(4) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$f''(4) = -\frac{1}{4}\left(\frac{1}{8}\right) = -\frac{1}{32}$

$$P_2(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{32}\cdot\frac{1}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

b) Use it to approximate $\sqrt{4.1}$

$$P_2(4.1) = 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2$$

$$\approx 2.02484375$$

c) Find an upper bound for $|R_2(4.1)|$

Recall $f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\quad f'''(z) = \frac{3}{8}z^{-5/2}$

$$|R_2(x)| = \left|\frac{f'''(z)}{3!}(x-c)^3\right|$$

where z is between 4 and 4.1

$$= \left|\frac{1}{3!}\cdot\frac{3}{8}z^{-5/2}(0.1)^3\right|$$

$$\leq \left|\frac{1}{3!}\cdot\frac{3}{8}(4)^{-5/2}(0.1)^3\right|$$

≤ 0.000002

d) Approximate $\sqrt{4.1}$, with error

$$2.02484375 - 0.000002 \leq \sqrt{4.1} \leq 2.02484375 + 0.000002$$