Test Average : 71%

9.5 Alternating Series Cont'd

Alternating Series

$\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

In an alternating series, a_n is the <u>unsigned</u> part of the term.

e.g. series $= 1 - \frac{1}{2} + \frac{1}{3} - \dots$

$\Rightarrow a_n = \frac{1}{n}$

If a series converges by the Alternating Series Test then $|R_N| \le a_{N+1}$.

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+1}$ converges by the Alternating Series Test.

Find N so that $|R_N| \le 0.005$

Plan: $a_{N+1} \le 0.005$

(Then $|R_N| \leq a_{N+1} \leq 0.005$)

$$a_n = \frac{1}{\sqrt{n}+1}$$

$$a_{N+1} = \frac{1}{\sqrt{N+1}+1}$$

$$\frac{1}{\sqrt{N+1}+1} \leq 0.005$$

$$1 \leq 0.005\left(\sqrt{N+1}+1\right)$$

$$\underset{200}{\cancel{\frac{1}{0.005}}} \leq \sqrt{N+1} + 1$$

$$199 \leq \sqrt{N+1}$$

$$39601 \leq N+1$$

$$39600 \leq N$$

$$\boxed{N \geq 39600}$$

Def

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating).

$\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ converges conditionally

if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ex: a) $1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges (Alternating)

$1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges (p-series)

$1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges conditionally

b) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ converges (Alternating)

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ converges (p-series)

$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ converges absolutely

Absolute Convergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating). If $\sum_{n=1}^{\infty} |a_n|$ converges

then $\sum_{n=1}^{\infty} a_n$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges.

Is it conditionally convergent or absolutely convergent?

Use Direct Comparison Test.

Let $a_n = \left|\frac{\cos n}{n^3}\right|$ and $b_n = \frac{1}{n^3}$

$$0 < \left|\frac{\cos n}{n^3}\right| \le \frac{1}{n^3} \quad \text{for } n \ge 1 \checkmark$$

(Note: $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\cos(\text{integer}) \neq 0$)

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3}\right|$ converges

$\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ is <u>absolutely convergent</u>.

9.6 The Ratio and Root Tests

Ratio Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

If $L<1$ then the series converges absolutely.
$L=1$ then the test is inconclusive.
$L>1$ then the series diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{3^n} \Longleftarrow |a_n| = \frac{n^3}{3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right|$$

$$= \left|\frac{(n+1)^3}{n^3} \cdot \frac{1}{3}\right|$$

$$= \left|\left(\frac{n+1}{n}\right)^3 \frac{1}{3}\right|$$

$$= \left|\left(\frac{1+\frac{1}{n}}{1}\right)^3 \frac{1}{3}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\left(\frac{1+\frac{1}{n}}{1}\right)^3 \frac{1}{3}\right|$$

$$= \frac{1}{3}$$

The series converges absolutely.

Recall (Section 5.6)

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

<u>Ex</u>: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \longleftarrow \quad a_n = \frac{n^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right|$$

$$= \left|\frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n}\right|$$

$$\boxed{(n+1)! = (n+1)n!}$$

$$= \left|\frac{1}{\cancel{n+1}} \cdot \frac{\cancel{(n+1)}(n+1)^n}{n^n}\right|$$

$$= \left|\left(\frac{n+1}{n}\right)^n\right|$$

$$= \left|\left(1+\frac{1}{n}\right)^n\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e \quad (>1)$$

The series diverges.

Root Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

If $L<1$ then the series converges absolutely.

$L=1$ then the test is inconclusive.

$L>1$ then the series diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+5}\right)^n \longleftarrow \quad a_n = \left(\frac{2n+3}{3n+5}\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+5}\right)^n}$$

$$= \frac{2n+3}{3n+5}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+5}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2}{3}$$

$$= \frac{2}{3}$$

The series converges absolutely.

9.6 #45

Use the Root Test to decide whether the series converges:

$$\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(2\sqrt[n]{n}+1\right)^n}$$

$$= 2\sqrt[n]{n}+1$$

Want $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 2\sqrt[n]{n}+1$

$$L = \lim_{n\to\infty} \sqrt[n]{n}$$

$$\ln L = \lim_{n\to\infty} \ln n^{1/n}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} \leftarrow \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1}$$

$$= 0$$

$$\boxed{\ln L = 0 \Rightarrow L = e^0 = 1}$$

Desired limit $= 2(1) + 1$

$= 3$

The series diverges.