

9.4 Comparisons of Series

Direct Comparison Test

Suppose $0 < a_n \leq b_n$ for all n .

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Also true if index starts at $n=0$ etc.

Recall: $\sum_{n=0}^{\infty} ar^n$ converges if $-1 < r < 1$
diverges if $|r| \geq 1$

Recall: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \leq 1$

Ex: Test $\sum_{n=1}^{\infty} \frac{1 + \ln n}{n}$ for convergence.

$$\frac{1 + \ln n}{n} \geq \frac{1}{n} \quad \text{for all } n$$

$$0 < \frac{1}{n} \leq \frac{1 + \ln n}{n} \quad \text{for all } n \quad \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1 + \ln n}{n}$ diverges.

$$\lim_{n \rightarrow \infty} \ln n = \infty$$

ASIDE

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 \quad (\text{L'Hôpital's Rule})$$

Ex: Test $\sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$ for convergence.

$$2^{(n^2)} \geq 2^n \quad \text{for all } n$$

$$\frac{1}{2^{(n^2)}} \leq \frac{1}{2^n} \quad \text{for all } n$$

$$0 < \frac{1}{2^{(n^2)}} \leq \frac{1}{2^n} \quad \text{for all } n \quad \checkmark$$

$\sum_{n=0}^{\infty} \frac{1}{2^n}$ (geometric $r = \frac{1}{2}$) converges

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$ converges.

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n .

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

Ex: Test $\sum_{n=1}^{\infty} \frac{1}{3n+5}$ for convergence.

Look at dominant terms: $\frac{1}{n}$

$\frac{1}{3n+5} > 0$ and $\frac{1}{n} > 0$ for all n ✓

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{3n+5}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{3n+5}$$

$$\stackrel{(\oplus)}{=} \lim_{n \rightarrow \infty} \frac{1}{3}$$

$$= \frac{1}{3} \quad \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n+5}$ diverges

Ex: Test $\sum_{n=1}^{\infty} \frac{3\sqrt{n}+2}{5n^2+3n+1}$ for convergence.

Look at dominant terms: $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$\frac{3\sqrt{n}+2}{5n^2+3n+1} > 0$ and $\frac{1}{n^{3/2}} > 0$ for all n ✓

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{3\sqrt{n} + 2}{5n^2 + 3n + 1}}{\left(\frac{1}{n^{3/2}}\right)} \\
&= \lim_{n \rightarrow \infty} \frac{3\sqrt{n} + 2}{5n^2 + 3n + 1} \cdot \frac{n^{3/2}}{1} \\
&= \lim_{n \rightarrow \infty} \frac{3n^2 + 2n^{3/2}}{5n^2 + 3n + 1} \\
&= \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n^{1/2}}}{5 + \frac{3}{n} + \frac{1}{n^2}} \\
&= \frac{3}{5} \quad \checkmark
\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} \frac{3\sqrt{n} + 2}{5n^2 + 3n + 1}$ Converges.

Direct Comparison Test works well when n is in the exponent or when $\ln n$ appears in the term.

Limit Comparison Test works well for ratios of polynomials.

Ex: Does the series converge or diverge?
Justify.

a) $\sum_{n=1}^{\infty} 3^{-n^2}$

Direct Comparison Test

$$3^{n^2} \geq 3^n \quad \text{for all } n$$

$$\frac{1}{3^{n^2}} \leq \frac{1}{3^n} \quad \parallel$$

$$0 < \frac{1}{3^{n^2}} \leq \frac{1}{3^n} \quad \parallel \quad \checkmark$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3^{n^2}} \text{ converges}$$

b) $\sum_{n=1}^{\infty} \frac{1 + |\sin n|}{\sqrt{n}}$

Direct Comparison

$$\frac{1}{\sqrt{n}} \leq \frac{1 + |\sin n|}{\sqrt{n}} \quad \text{for all } n$$

$$0 < \frac{1}{\sqrt{n}} \leq \frac{1 + |\sin n|}{\sqrt{n}} \quad \parallel \quad \checkmark$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1 + |\sin n|}{\sqrt{n}} \text{ diverges}$$

$$c) \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n^2+1}}$$

Limit Comparison

$$\text{Dominant terms} = \frac{1}{n\sqrt{n^2}}$$

$$= \frac{1}{n^2}$$

$$\frac{2}{n\sqrt{n^2+1}} > 0 \quad \text{and} \quad \frac{1}{n^2} > 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{n\sqrt{n^2+1}} \right)}{\left(\frac{1}{n^2} \right)}$$

$$= \lim_{n \rightarrow \infty} \frac{2n^2}{n\sqrt{n^2+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2+1}}$$

$$= \lim_{n \rightarrow \infty} 2 \sqrt{\frac{n^2}{n^2+1}}$$

$$= \lim_{n \rightarrow \infty} 2 \sqrt{1 + \frac{1}{n^2}}$$

$$= 2 \quad \checkmark$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n^2+1}}$ Converges