Final Exam
Thurs Dec 14
1:30 – 4:30 pm
TEC 175

9.3 The Integral Test and p-series

Def

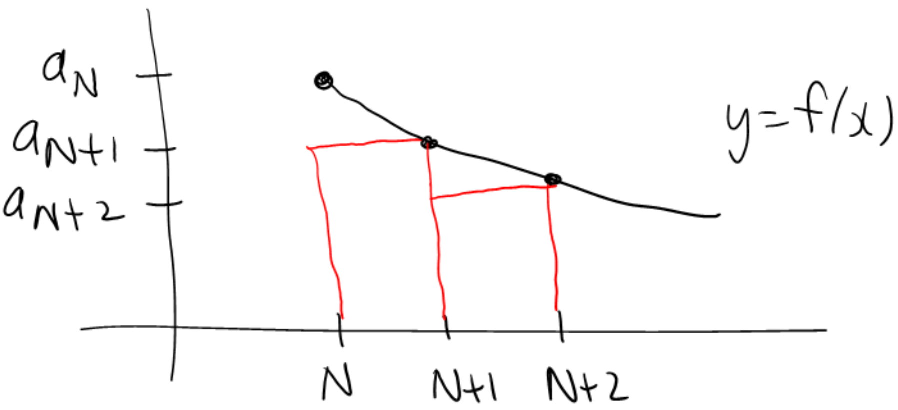
$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N} a_n}_{\substack{\text{partial sum} \\ S_N}} + \underbrace{\sum_{n=N+1}^{\infty} a_n}_{\substack{\text{remainder or error} \\ R_N}}$$

FACT

Suppose $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test.

Then $R_N \le \int_N^{\infty} f(x)\,dx$

Why?



$$R_N \le \int_N^{\infty} f(x)\,dx$$

sum of areas of red rectangles

area under curve

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges by the Integral Test.

a) Find S_{10}

$$S_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \frac{1}{512} + \frac{1}{729} + \frac{1}{1000}$$

$$\approx 1.19753199$$

b) Find an upper bound for the error R_{10}

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$\leq \lim_{b \to \infty} \int_{10}^{b} x^{-3}\,dx$$

$$\leq \lim_{b \to \infty} -\frac{1}{2}x^{-2} \Big|_{10}^{b}$$

$$\leq \lim_{b \to \infty} \frac{-1}{2}b^{-2} + \frac{1}{2}(10)^{-2}$$

$$\leq 0.005$$

c) Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ using S_{10} and R_{10}.

$$S_{10} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq S_{10} + R_{10}$$

$$1.19753199 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.19753199 + 0.005$$

d) Find N so that $R_N \leq 0.0005$

Plan: Calculate $\int_N^{\infty} \frac{1}{x^3}\,dx$

Solve $\int_N^{\infty} \frac{1}{x^3}\,dx \leq 0.0005$

$$\int_N^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_N^b x^{-3}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{2} x^{-2} \Big|_N^b$$

$$= \lim_{b\to\infty} -\frac{1}{2} b^{-2} + \frac{1}{2} N^{-2}$$

$$= \frac{1}{2} N^{-2}$$

$$\frac{1}{2} N^{-2} \leq 0.0005$$

$$\frac{1}{2} \leq 0.0005 N^2$$

$$\frac{1}{2(0.0005)} \leq N^2$$

Take square roots:

$$\sqrt{\frac{1}{2(0.0005)}} \leq N$$

$N \geq 31.6$

$\boxed{N \geq 32}$

R Code for Partial Sums on website

Ex: Does the series converge or diverge?

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-4n}{n^2+1}$

DIVERGES

$\displaystyle\lim_{n\to\infty} \frac{3n^2-4n}{n^2+1} \overset{H}{=} \lim_{n\to\infty} \frac{6n-4}{2n} \overset{H}{=} \lim_{n\to\infty} \frac{6}{2} \neq 0$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{9}{8}\right)^n$

DIVERGES Geometric $r=\frac{9}{8}$

c) $\displaystyle\sum_{n=\infty}^{\infty} \frac{1}{7}\left(\frac{8}{9}\right)^n$

CONVERGES Geometric $r=\frac{8}{9}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.05}}$ CONVERGES p-series with $p>1$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$ DIVERGES p-series with $p\leq 1$

f) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2}{n+7} - \frac{2}{n+8}\right] = \left(\frac{2}{8}-\frac{2}{9}\right)+\left(\frac{2}{9}-\frac{2}{10}\right)+\ldots$

$$= \frac{2}{8} - \lim_{N\to\infty} \frac{2}{N+8}$$

$$= \frac{2}{8} - 0$$

$$= \frac{1}{4}$$

CONVERGES (Telescoping Series and $\lim_{N\to\infty} b_N$ exists)