9.2 Series and Convergence Cont'd

FACT

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge

then $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$ (c: any real #).

Ex: Find $\sum_{n=2}^{\infty} \frac{12 + 4(2^{n+1})}{5^n}$

$$= \sum_{n=2}^{\infty} \frac{12}{5^n} + \sum_{n=2}^{\infty} \frac{4(2^{n+1})}{5^n}$$

GEOMETRIC $a = 12/25$ $r = 1/5$

GEOMETRIC $a = \frac{32}{25}$ $r = \frac{2}{5}$

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r} \text{ if } -1 < r < 1$$

"GEOMETRIC SERIES"

$$= \frac{\left(\frac{12}{25}\right)}{\left(\frac{4}{5}\right)} + \frac{\left(\frac{32}{25}\right)}{\left(\frac{3}{5}\right)}$$

$$= \frac{12}{25} \times \frac{5}{4} + \frac{32}{25} \times \frac{5}{3}$$

$$= \frac{3}{5} + \frac{32}{15}$$

$$= \frac{41}{15}$$

A series of the form

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - b_2 + b_2 - b_3 + \dots$$

is a <u>telescoping series</u>.

Partial sums:

$$S_1 = b_1 - b_2$$

$$S_2 = b_1 - b_2 + b_2 - b_3 = b_1 - b_3$$

$$S_3 = b_1 - b_4$$

$$S_N = b_1 - b_{N+1}$$

$$S = \lim_{N \to \infty} S_N$$

$$= b_1 - \lim_{N \to \infty} b_N$$

Fact

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{N \to \infty} b_N$$

if the limit exists and is a real number.

The series diverges otherwise.

<u>Ex</u>: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Use partial fractions to write as a telescoping series.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

Sub $n=0$: $\quad 1 = A$

$n=-1$: $\quad 1 = -B \Rightarrow B = -1$

$$\text{Series} = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots$$

$$= b_1 - \lim_{N\to\infty} b_N$$

$$= 1 - \lim_{N\to\infty} \frac{1}{N}$$

$$= 1$$

Fact: n^{th} Term Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: Consider $\sum_{n=1}^{\infty} \frac{3n+1}{5n+1}$

$$\lim_{n\to\infty} \frac{3n+1}{5n+1} \leftarrow \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{3}{5}$$

$$= \frac{3}{5}$$

$$\neq 0$$

Series diverges.

Caution: If $\lim_{n\to\infty} a_n = 0$ then the series <u>may converge</u> or <u>may diverge</u>.

Quick Ex:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{Converges}$$

and both have $\lim_{n\to\infty} a_n = 0.$

(Details in Section 9.3)

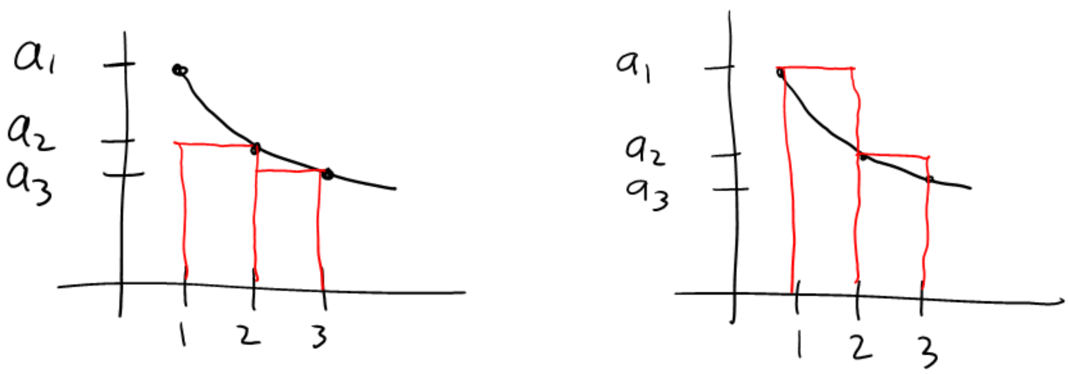
9.3 The Integral Test and p-Series

Integral Test

If f is continuous, positive and decreasing on $[1,\infty)$ and $a_n = f(n)$ for $n = 1, 2, 3, \ldots$ then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

both converge or both diverge.



$$a_2 + a_3 + \ldots \le \int_1^{\infty} f(x)\,dx \le a_1 + a_2 + a_3 + \ldots$$

$\sum_{n=1}^{\infty} a_n$ is a finite number

$\Longleftrightarrow$ $\int_1^{\infty} f(x)\,dx$ is a finite number.

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

$f(x) = \frac{1}{x^2}$

$f(x)$ is continuous on $[1, \infty)$ ✓

$f(x)$ " positive " ✓

$f'(x) = -2x^{-3}$

$f'(x) < 0$ on $[1, \infty)$

$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx$$

$$= \lim_{b\to\infty} -x^{-1} \Big|_1^{b}$$

$$= \lim_{b\to\infty} -\frac{1}{b} + 1$$

$$= 1$$

The series converges.

Caution: No info about the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

$$f(x) = \frac{1}{x}$$

$f(x)$ is continuous on $[1, \infty)$ ✓

" positive " ✓

$f'(x) = -x^{-2} < 0$ on $[1, \infty)$ ✓

$$\int_1^{\infty} \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b - \ln 1$$

$$= \infty$$

The series diverges.

FACT

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \le 1$

Terminology: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a <u>p-series</u>.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is called the <u>harmonic series</u>.

Recap:

$$\sum_{n=1}^{\infty} 5(0.2)^n = 1.25$$ (Geometric)

$$\sum_{n=1}^{\infty} \left(\frac{3}{n} - \frac{3}{n+1}\right) = 3$$ (Telescoping)

$\sum_{n=1}^{\infty} n$ diverges (nth term test)

because $\lim_{n\to\infty} n \neq 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series)