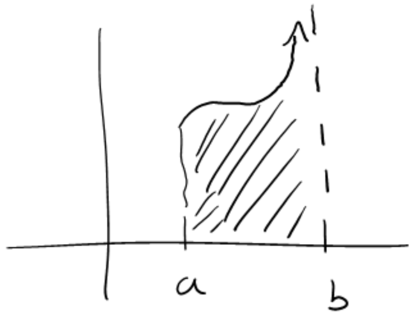
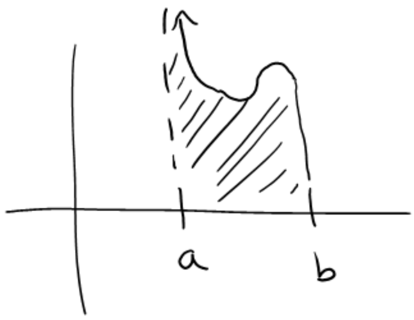
8.8 Improper Integrals Cont'd

See handout on website.
Read page two.

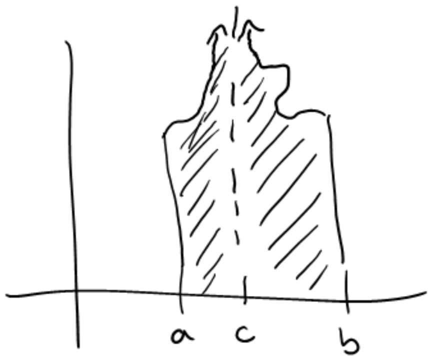
Picture for Fact 4



Picture for Fact 5



Picture for Fact 6



Ex: Evaluate or show that it diverges.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \int_t^5 (x-2)^{-1/2}\,dx$$

$$= \lim_{t\to 2^+} 2(x-2)^{1/2} \Big|_t^5$$

$$= \lim_{t\to 2^+} 2\sqrt{3} - 2\sqrt{t-2}$$

$$= 2\sqrt{3}$$

Ex: Evaluate or show that it diverges.

$$\int_0^3 \frac{1}{(x-3)^2}\,dx$$

$$= \lim_{t\to 3^-} \int_0^t (x-3)^{-2}\,dx$$

$$= \lim_{t\to 3^-} -(x-3)^{-1} \Big|_0^t$$

$$= \lim_{t\to 3^-} \frac{-1}{t-3} + \frac{1}{-3}$$

$$= +\infty - \frac{1}{3}$$

$$= \infty$$

The integral diverges.

Ex: Evaluate or show that it diverges.

$$\int_0^3 \frac{1}{x-1}\,dx$$

Caution: Asymptote at $x=1$

$$\int_0^3 \frac{1}{x-1}\,dx = \underbrace{\int_0^1 \frac{1}{x-1}\,dx}_{I_1} + \underbrace{\int_1^3 \frac{1}{x-1}\,dx}_{I_2}$$

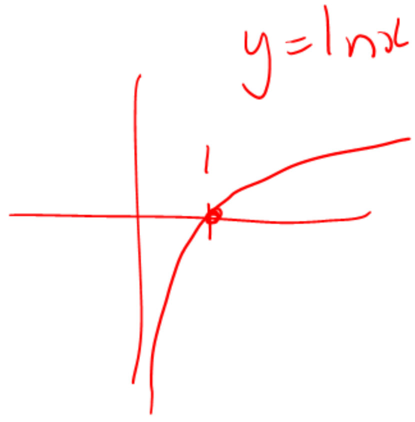
$$I_1 = \int_0^1 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \ln|x-1| \Big|_0^t$$

$$= \lim_{t\to 1^-} \ln|t-1| - \cancelto{0}{\ln 1}$$

$$= -\infty$$



I_1 diverges

$\Rightarrow$ Integral diverges.

Extra : Evaluate or show that it diverges.

$$\int_0^3 (x-2)^{-2/3}\,dx$$

Caution: Asymptote at $x=2$

$$\int_0^3 (x-2)^{-2/3}\,dx = \underbrace{\int_0^2 (x-2)^{-2/3}\,dx}_{I_1} + \underbrace{\int_2^3 (x-2)^{-2/3}\,dx}_{I_2}$$

$$I_1 = \int_0^2 (x-2)^{-2/3}\,dx$$

$$= \lim_{t\to 2^-} \int_0^t (x-2)^{-2/3}\,dx$$

$$= \lim_{t\to 2^-} 3(x-2)^{1/3} \Big|_0^t$$

$$= \lim_{t\to 2^-} 3(t-2)^{1/3} - 3(-2)^{1/3}$$

$$= 0 + 3(2)^{1/3}$$

$$= 3\cdot\sqrt[3]{2}$$

$$I_2 = \int_2^3 (x-2)^{-2/3}\,dx$$

$$= \lim_{t\to 2^+} \int_t^3 (x-2)^{-2/3}\,dx$$

$$= \lim_{t\to 2^+} 3(x-2)^{1/3} \Big|_t^3$$

$$= \lim_{t \to 2^+} 3 - 3(t-2)^{1/3}$$

$$= 3$$

$$\text{Integral} = I_1 + I_2$$
$$= 3 \cdot \sqrt[3]{2} + 3$$

9.1 Sequences

A sequence is an infinite ordered list of numbers.

Notation:

$$a_1, a_2, a_3, \ldots$$

or $\{ a_n \}_{n=1}^{\infty}$

A sequence could begin at $n=0$ or any other value.

Ex: Find the first three terms

$$\left\{ \frac{(-1)^n}{2^n} \right\}_{n=0}^{\infty}$$

$a_0 = \frac{(-1)^0}{2^0} = 1$, $a_1 = \frac{-1}{2}$, $a_2 = \frac{1}{4}$

Ex: $a_0 = 0$, $a_1 = 1$
$a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$
Find the next 3 terms.

$(n=0)$ $a_2 = a_1 + a_0 = 1$
$(n=1)$ $a_3 = a_2 + a_1 = 2$
$(n=2)$ $a_4 = a_3 + a_2 = 3$

If $\lim_{n\to\infty} a_n$ exists and is a finite number then the sequence converges. Otherwise it diverges.

Ex: Find the sequence's limit, if possible.

a) $\left\{ \frac{1}{3n+1} \right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{1}{3n+1}$$
$$= 0$$
Sequence converges to 0.

b) $\{ n^2 \}_{n=1}^{\infty}$

$$\lim_{n\to\infty} n^2$$
$$= \infty$$
Sequence diverges.

c) $\{ (-1)^n \}_{n=1}^{\infty}$

$\lim_{n\to\infty} (-1)^n$ is undefined (oscillates)

Sequence diverges.

d) $\left\{ \frac{3n}{\sqrt{n^2+1}} \right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} 3\sqrt{\frac{n^2}{n^2+1}}$$

$$= \lim_{n\to\infty} 3\sqrt{\frac{1}{1+\frac{1}{n^2}}}$$

$$= 3\sqrt{\frac{1}{1+0}}$$

$$= 3$$

Sequence converges to 3.

We can use l'Hôpital's Rule on sequences.

Ex: $\lim_{n\to\infty} \frac{2n}{3n+1}$ $\frac{\infty}{\infty}$

$$\overset{H}{=} \lim_{n\to\infty} \frac{2}{3}$$

$$= \frac{2}{3}$$

$\left\{ \frac{2n}{3n+1} \right\}_{n=1}^{\infty}$ converges to $\frac{2}{3}$.

Squeeze Theorem

If $a_n \le b_n \le c_n$ for all n

and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$.

Ex: Find the limit of $\left\{ \frac{\cos n}{n} \right\}_{n=1}^{\infty}$

$$\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

and $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

Therefore $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

$$n! = n(n-1)(n-2)\cdots 1$$

"n factorial"

$$5! = 5(4)(3)(2)(1) = 120$$

$$3! = 3(2)(1) = 6$$

$$2! = 2(1) = 2$$

$$1! = 1$$

$0! = 1$ by definition

Ex: Simplify $\dfrac{(n+1)!}{(n-1)!}$

$$= \frac{(n+1)(n)\cancel{(n-1)(n-2)\cdots}}{1\,\cancel{(n-1)(n-2)\cdots}}$$

$$= (n+1)n$$