5.6 Indeterminate Forms Cont'd

We'll use logarithms to handle the forms 0^0, ∞^0 and 1^∞.

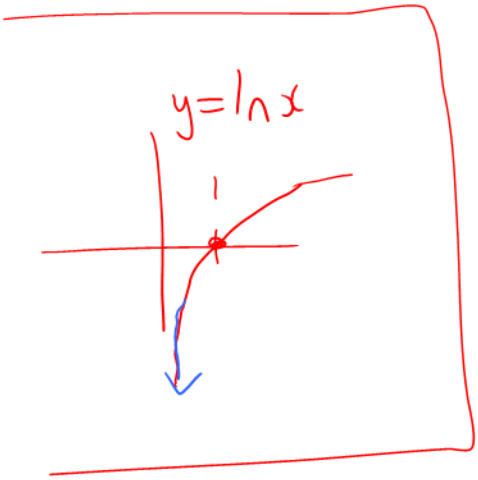
<u>Ex</u>: $\lim_{x\to 0^+} x^x$

Let $L = \lim_{x\to 0^+} x^x$

$$\ln L = \ln\left(\lim_{x\to 0^+} x^x\right)$$

$$= \lim_{x\to 0^+} \ln x^x$$

$$= \lim_{x\to 0^+} x \ln x \leftarrow \quad 0(-\infty)$$



$$= \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \leftarrow \quad \frac{-\infty}{\infty}$$

$$\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to 0^+} \cancel{-x^2\left(\frac{1}{x}\right)} \; -x$$

$$= 0$$

$$\ln L = 0$$

$$L = e^{\ln L} = e^{0} = 1$$

Ex: $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

let $L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

$$\ln L = \lim_{x \to \infty} \ln \left(1 + \frac{1}{x}\right)^{x}$$

$$= \lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) \leftarrow \infty(0)$$

$$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \leftarrow \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{\left(1 + \frac{1}{x}\right)} \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= 1$$

$$\ln L = 1$$

$$L = e^{\ln L} = e^{1} = e$$

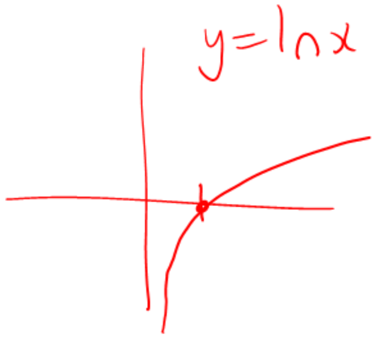
Ex: $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \tan x^{\cos x}$

Let $\displaystyle L = \lim_{x\to\frac{\pi}{2}^-} \tan x^{\cos x}$

$$\ln L = \lim_{x\to\frac{\pi}{2}^-} \ln \tan x^{\cos x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} (\cos x)\ln \tan x$$

$0(\infty)$



$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\ln \tan x}{\sec x}$$

$\frac{\infty}{\infty}$

$$\overset{H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\tan x}(\sec^2 x)}{\sec x \tan x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cancel{\cos x}} \frac{\cos^{\cancel{2}} x}{\sin^2 x}$$

$$= 0$$

$\ln L = 0$

$L = e^{\ln L} = e^0 = 1$

8.8 Improper Integrals

See handout on website.
Read page 1.

Ex: Evaluate or show that it diverges.

a) $\int_1^\infty \frac{1}{x}\,dx$

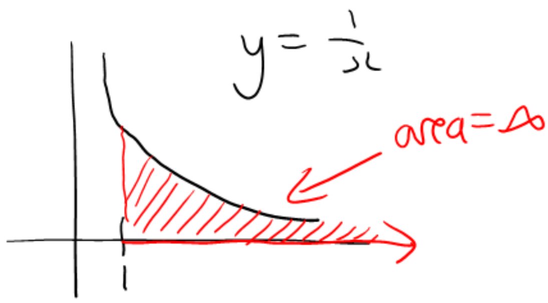
$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \left[\ln|x|\right]_1^b$

$= \lim_{b\to\infty} \ln b$

$= \infty$

The integral diverges.

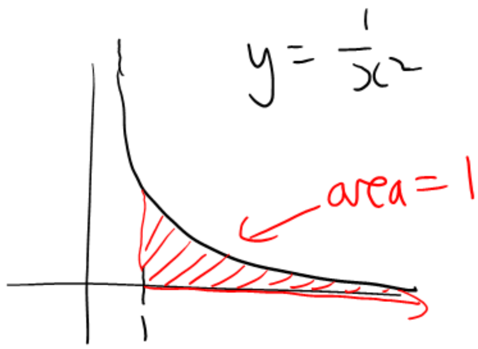


b) $\int_1^\infty \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \left[-x^{-1}\right]_1^b$

$= \lim_{b\to\infty} -\frac{1}{b} + 1$



$= \qquad 1$

FACT

Let p be a real number.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & , \text{ if } p>1 \\ \infty & , \text{ if } p \le 1 \end{cases}$$

Ex: Evaluate or show that it diverges.

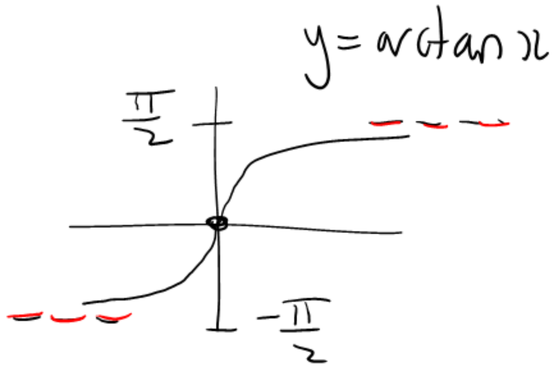
a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$= \lim_{b\to\infty} \arctan b$$

$$= \frac{\pi}{2}$$



b) $\displaystyle\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$

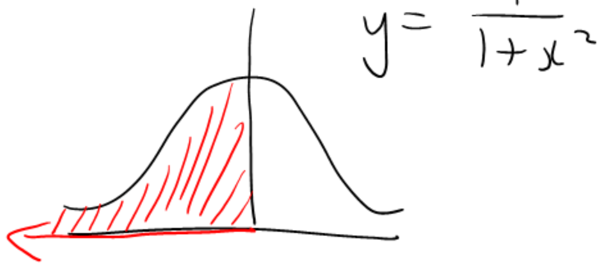
$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \left[\arctan x\right]_a^0$$

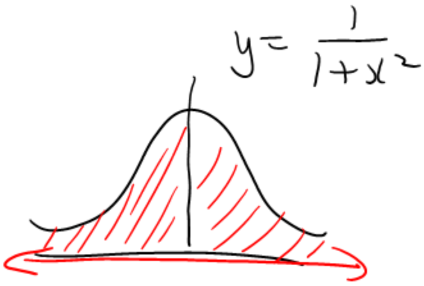
$$= \lim_{a \to -\infty} \quad -\arctan a$$

$$= \quad -\left(-\frac{\pi}{2}\right)$$

$$= \quad \frac{\pi}{2}$$



c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

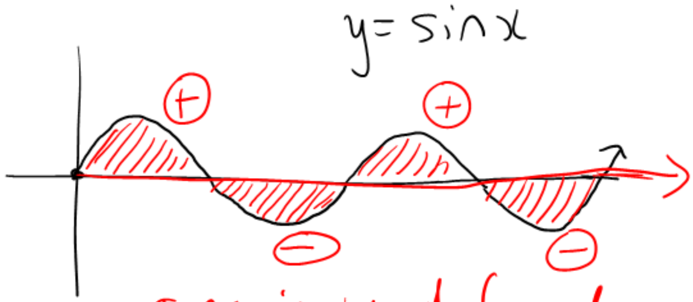


$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\, dx + \int_{0}^{\infty} \frac{1}{1+x^2}\, dx$$

$$= \quad \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \quad \pi$$

d) $\displaystyle\int_0^{\infty} \sin x \, dx$



$$= \lim_{b \to \infty} \int_0^b \sin x \, dx$$

$$= \lim_{b \to \infty} \left[-\cos x\right]_0^b$$

$$= \lim_{b \to \infty} \quad -\cos b + 1$$

$= \text{undefined}$ (cos b is oscillating as $b \to \infty$)

The integral diverges.

e) $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx$

$$= \int_{-\infty}^{0} \sin x \, dx + \underbrace{\int_{0}^{\infty} \sin x \, dx}_{\text{diverges}}$$

Therefore $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx$ diverges.

ASIDE

If $f(x) < 0$ then $\displaystyle\int_a^b f(x)\,dx$ may be < 0.