5.6 Indeterminate Forms

Indeterminate forms:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0(\infty)$, $\infty-\infty$, 0^0, ∞^0, 1^∞

The limit may or may not exist.

L'Hôpital's Rule

Suppose $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ as long as the second limit exists (or is $\pm\infty$).

Note:

$x\to a$ also includes $x\to a^-$, $x\to a^+$,

$x\to\infty$, $x\to -\infty$

Ex: $\lim_{x\to\infty} \frac{x^2+1}{3x^2+5x+4}$

The form is $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{x^2+1}{3x^2+5x+4} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{6x+5}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{2}{6}$$

$$= \frac{1}{3}$$

Note: The form must be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$ each time l'Hôpital's Rule is applied.

Ex: $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

The form is $\frac{0}{0}$.

$$\overset{(H)}{=} \lim_{x\to 0} \frac{\cos x}{1}$$

$$= 1$$

Ex: $\displaystyle\lim_{x\to\infty} \frac{3x^2+7x+1}{e^{2x}}$

The form is $\frac{\infty}{\infty}$.

$$\overset{(H)}{=} \lim_{x\to\infty} \frac{6x+7}{2e^{2x}} \qquad \frac{\infty}{\infty}$$

$$\overset{(H)}{=} \lim_{x\to\infty} \frac{6}{4e^{2x}}$$

$$= 0$$

FACT

$\displaystyle\lim_{x\to\infty} \frac{P_n(x)}{e^{ax}} = 0$ where

$P_n(x) =$ polynomial of degree n

and $a > 0$

Ex: $\lim_{x \to 2} \dfrac{1-\cos(x^3-8)}{(x-2)^2}$

The form is $\dfrac{0}{0}$

$$\overset{H}{=} \lim_{x \to 2} \frac{3x^2 \sin(x^3-8)}{2(x-2)}$$

$\dfrac{0}{0}$

$$\overset{H}{=} \lim_{x \to 2} \frac{9x^4 \cos(x^3-8) + 6x\sin(x^3-8)}{2(1)}$$

$$= \frac{144(1)+0}{2}$$

$$= 72$$

Ex: $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

The form is $\infty(0)$. Rewrite.

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

The form is $\dfrac{0}{0}$

$$\overset{H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \cos 0$$

$$= 1$$

Ex: $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

The form is $\infty - \infty$. Rewrite.

$$= \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x}$$

$\frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{(x-1)\frac{1}{x} + \ln x}$$

$\frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x^2}}{(x-1)\left(\frac{-1}{x^2}\right) + \frac{1}{x} + \frac{1}{x}}$$

$$= \frac{1}{2}$$