# 8.5 Partial Fractions Cont'd

Ex: $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x+?)(x+?)$$
$$= x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

sub $x=0$: $\quad -1 = A(-2) \Rightarrow A = \frac{1}{2}$

$x=-2$: $\quad -1 = C(-2)(-5) \Rightarrow C = -\frac{1}{10}$

$x=\frac{1}{2}$: $\quad \frac{1}{4} = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$

$$\text{Integral} = \int \left[\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{(2x-1)} - \frac{1}{10}\frac{1}{(x+2)}\right]dx$$

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C}$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

Note: If degree(numerator) $\geq$ degree(denominator), then need long division.

Ex: $\displaystyle\int \frac{2x^3-3x^2-3x-10}{x^2-2x-3}\,dx$

$$
\begin{array}{r}
2x+1 \phantom{-3x-10} \\
(x^2-2x-3)\,\overline{)\,2x^3-3x^2-3x-10} \\
-(2x^3-4x^2-6x) \phantom{-10} \\
\hline
x^2+3x-10 \\
-(x^2-2x-3) \\
\hline
5x-7
\end{array}
$$

Integrand $= 2x+1+\dfrac{5x-7}{x^2-2x-3}$

$$\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$$

$$\frac{5x-7}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$\boxed{5x-7 = A(x+1)+B(x-3)}$$

$x=-1$: $\quad -12 = B(-4) \Rightarrow B = 3$

$x=3$: $\quad 8 = A(4) \Rightarrow A = 2$

$$\text{Integrand} = 2x+1 + \frac{2}{x-3} + \frac{3}{x+1}$$

$$\text{Integral} = x^2 + x + 2\ln|x-3| + 3\ln|x+1| + C$$

We've seen denominators consisting of <u>distinct</u> <u>linear</u> factors.

## Repeated Linear Factors

FACT

$$\frac{\text{polynomial}}{(x+1)(x+5)^2} = \frac{A}{x+1} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

$$\frac{\text{polynomial}}{(x+1)(x+5)^n} = \frac{A}{x+1} + \frac{A_1}{(x+5)} + \frac{A_2}{(x+5)^2} + \frac{A_3}{(x+5)^3} + \dots + \frac{A_n}{(x+5)^n}$$

(assuming long division is not required)

<u>Ex</u>: $\displaystyle\int \frac{2x}{(x+1)(x+2)^3}\,dx$

$$\frac{2x}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$2x = A(x+2)^3 + B(x+1)(x+2)^2 + C(x+1)(x+2) + D(x+1)$$

$x=-1$: $\quad -2 = A(1) \Rightarrow A = -2$

$x=-2$: $\quad -4 = D(-1) \Rightarrow D = 4$

$x^3$ coefficient: $\quad 0 = A + B \Rightarrow B = 2$

Sub any number:

$x=0$: $\quad 0 = A(8) + 4B + C(2) + D(1)$

$$0 = -16 + 8 + 2C + 4$$

$$4 = 2C$$

$$C = 2$$

$$\text{Integral} = \int\left[\frac{-2}{x+1} + \frac{2}{x+2} + \frac{2}{(x+2)^2} + \frac{4}{(x+2)^3}\right]dx$$

$$= -2\ln|x+1| + 2\ln|x+2| - 2(x+2)^{-1} - 2(x+2)^{-2} + C$$

## Irreducible Quadratic Factors

$ax^2+bx+c$ is irreducible (can't be factored)

if $b^2-4ac<0$

Ex: $x^2+4$ is irreducible

$x^2+4x+13$ "

FACT

$$\frac{\text{polynomial}}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

$$\frac{\text{polynomial}}{(x+3)(x^2+4)^2} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

(assuming long division is not required)

Note: Every polynomial can be factored into linear and irreducible quadratic factors.

Ex:
$$\int \frac{2x+1}{x^4+3x^2-4}\,dx$$

$$x^4+3x^2-4 = (x^2-1)(x^2+4)$$
$$= (x-1)(x+1)(x^2+4)$$

$$\frac{2x+1}{(x-1)(x+1)(x^2+4)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+4}$$

$$2x+1 = A(x+1)(x^2+4) + B(x-1)(x^2+4) + (Cx+D)(x-1)(x+1)$$

$x=-1$: $\quad -1 = B(-2)(5) \quad \Rightarrow \quad B = \frac{1}{10}$

$x=1$: $\quad 3 = A(2)(5) \quad \Rightarrow \quad A = \frac{3}{10}$

$x^3$ Coefficient:

$$0 = A + B + C$$
$$0 = \frac{3}{10} + \frac{1}{10} + C$$
$$C = \frac{-4}{10} = \frac{-2}{5}$$

Sub any value:
$x = 0$:

$$1 = 4A - 4B + D(-1)(1)$$
$$1 = \frac{12}{10} - \frac{4}{10} - D$$
$$D = \frac{12}{10} - \frac{4}{10} \quad \frac{-10}{10}$$
$$D = \frac{-2}{10} = \frac{-1}{5}$$

$$\text{Integral} = \int \left[ \frac{3}{10} \cdot \frac{1}{x-1} + \frac{1}{10} \cdot \frac{1}{x+1} + \frac{-\frac{2}{5}x - \frac{1}{5}}{x^2+4} \right] dx$$

$$= \int \left[ \frac{3}{10} \frac{1}{(x-1)} + \frac{1}{10} \frac{1}{(x+1)} - \frac{2}{5} \frac{x}{x^2+4} - \frac{1}{5} \cdot \frac{1}{x^2+4} \right] dx$$

$$= \frac{3}{10} \ln|x-1| + \frac{1}{10} \ln|x+1| - \frac{1}{5} \ln|x^2+4|$$
$$- \frac{1}{10} \tan^{-1} \frac{x}{2} + C$$