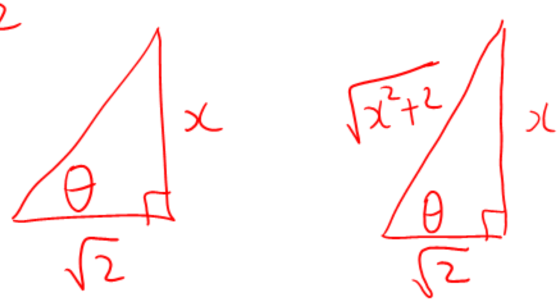
8.4 Trig Substitution Cont'd

Ex: $\displaystyle\int \frac{dx}{\sqrt{x^2+2}^{\,3}}$

$x = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta\, d\theta$

$\frac{x}{\sqrt{2}} = \tan\theta$



$$\frac{\sqrt{x^2+2}}{\sqrt{2}} = \sec\theta$$

$$\sqrt{x^2+2} = \sqrt{2}\sec\theta$$

$$\text{Integral} = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{(\sqrt{2}\sec\theta)^3}$$

$$= \frac{1}{2}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{2}\int \cos\theta\, d\theta$$

$$= \frac{1}{2}\sin\theta + C$$

$$= \frac{1}{2}\,\frac{x}{\sqrt{x^2+2}} + C$$

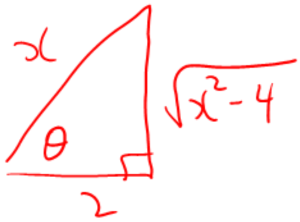
Ex:

$$\int_3^5 \frac{dx}{x^2\sqrt{x^2-4}}$$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$$\boxed{\begin{gathered} \frac{x}{2} = \sec\theta \\ \frac{\sqrt{x^2-4}}{2} = \tan\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{gathered}}$$



$$\begin{cases} a^2+b^2=c^2 \\ 4+b^2=x^2 \\ b^2=x^2-4 \\ b=\sqrt{x^2-4} \end{cases}$$

$$\text{Integral} = \int_{x=3}^{x=5} \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^{\cancel{2}}\theta\,(2\tan\theta)}$$

$$= \frac{1}{4}\int_{x=3}^{x=5} \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int_{x=3}^{x=5} \cos\theta\, d\theta$$

$$= \frac{1}{4}\Big[\sin\theta\Big]_{x=3}^{x=5}$$

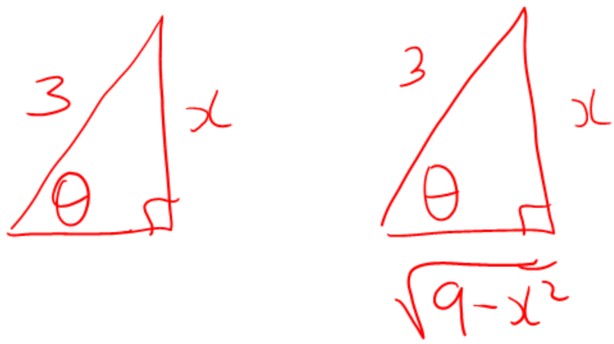
$$= \frac{1}{4}\left[\frac{\sqrt{x^2-4}}{x}\right]_3^5$$

$$= \frac{1}{4}\left[\frac{\sqrt{21}}{5} - \frac{\sqrt{5}}{3}\right]$$

Ex: $\int \sqrt{9-x^2}\,dx$

$$x = 3\sin\theta$$
$$dx = 3\cos\theta\, d\theta$$

$$\frac{x}{3} = \sin\theta$$



$$\frac{\sqrt{9-x^2}}{3} = \cos\theta$$

$$\boxed{\sqrt{9-x^2} = 3\cos\theta}$$

$$\text{Integral} = \int (3\cos\theta)(3\cos\theta\, d\theta)$$

$$= 9\int \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\overset{2\sin\theta\cos\theta}{\cancel{\sin 2\theta}}}{2}\right] + C$$

$$= \frac{9}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$\theta = \sin^{-1}\frac{x}{3}$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} + \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right] + C$$

<u>Ex</u>: $\displaystyle\int \frac{dx}{\sqrt{4x^2+9}}$

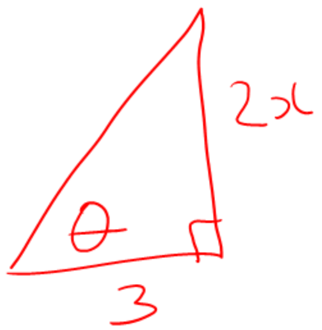
$$= \int \frac{dx}{\sqrt{(2x)^2+3^2}}$$

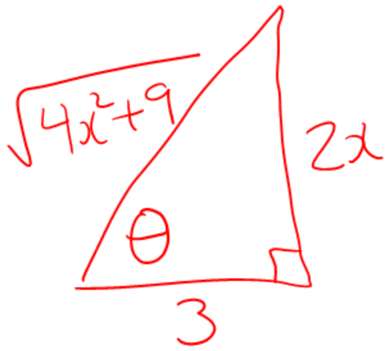
$$2x = 3\tan\theta$$

$$x = \frac{3}{2}\tan\theta$$

$$dx = \frac{3}{2}\sec^2\theta \, d\theta$$

$$\frac{2x}{3} = \tan\theta$$





$$\frac{\sqrt{4x^2+9}}{3} = \sec\theta$$

$$\sqrt{4x^2+9} = 3\sec\theta$$

$$\text{Integral} = \int \frac{\frac{3}{2}\sec^2\theta \, d\theta}{3\sec\theta}$$

$$= \frac{1}{2}\int \sec\theta \, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3} \right| + C$$

8.5 Partial Fractions

$$\frac{7x+1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

"partial fraction expansion"

Multiply by $(x+3)(x-1)$:

$$\boxed{7x+1 = A(x-1) + B(x+3)}$$

Method 1: SUB

Sub $x=1$: $\quad 8 = B(4) \implies B=2$

$x=-3$: $\quad -20 = A(-4) \implies A=5$

Method 2: MATCH COEFFICIENTS

$$7x+1 = Ax - A + Bx + 3B$$

$$7x + 1 = (A+B)x + (-A+3B)$$

x coefficient: $A + B = 7$ ①

constant: $-A + 3B = 1$ ②

①+②: $4B = 8$

$B = 2$

$B = 2 \to$ ①: $A + 2 = 7$

$A = 5$

Conclude $\dfrac{7x+1}{(x+3)(x-1)} = \dfrac{5}{x+3} + \dfrac{2}{x-1}$

Quick Ex:

$$\int \frac{7x+1}{(x+3)(x-1)}\,dx$$

$\vdots$

$$= \int \left[\frac{5}{x+3} + \frac{2}{x-1} \right] dx$$

$$= 5\ln|x+3| + 2\ln|x-1| + C$$

$$\boxed{\text{Shortcut} \quad \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C}$$