8.1 Basic Trig Integrals Cont'd

Ex: Find $\int \frac{1}{\sqrt{x}\sin\sqrt{x}}\,dx$

$$= \int \frac{\csc\sqrt{x}}{\sqrt{x}}\,dx$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \tfrac{1}{2}x^{-1/2}\,dx \\ 2\,du &= \frac{dx}{\sqrt{x}} \end{aligned}}$$

$$= 2\int \csc u\,du$$

$$= -2\ln|\csc u + \cot u| + C$$

$$= -2\ln|\csc\sqrt{x} + \cot\sqrt{x}| + C$$

Ex: $\int \frac{e^x \cos e^x}{\sin e^x}\,dx$

$$= \int e^x \cot e^x\,dx$$

$$\boxed{\begin{aligned} u &= e^{x} \\ du &= e^{x}\,dx \end{aligned}}$$

$$= \int \cot u \, du$$

$$= -\ln|\csc u| + C$$

$$= -\ln|\csc e^{x}| + C$$

<u>Ex</u>: $\displaystyle\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$= \int \sec u \tan u \, du$$

$$= \sec u + C$$

$$= \sec(\ln x) + C$$

<u>Recall</u>

$$\int x^{n}\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

Mixed Practice

Ex: Find $\int \frac{(1+\ln x)^3}{x} \, dx$

$$\boxed{\begin{aligned} u &= 1 + \ln x \\ du &= \frac{1}{x} \, dx \end{aligned}}$$

$$= \int u^3 \, du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{1}{4}(1+\ln x)^4 + C$$

Ex: Find

a) $\int \frac{x}{9+x^2}\,dx$

$$\boxed{\begin{aligned} u &= 9+x^2 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|9+x^2| + C$$

b) $\int \frac{x}{9+x^4}\,dx$

$$= \int \frac{x}{9+(x^2)^2}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{9+u^2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\frac{u}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\frac{x^2}{3} + C$$

<u>Ex</u>: $\int (\sec^2 x) e^{\tan x}\, dx$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array}}$$

$$= \int e^u\, du$$

$$= e^u + C$$

$$= e^{\tan x} + C$$

<u>Ex</u>: Find $\int \frac{e^{4x}}{\sqrt{9 - e^{8x}}}\, dx$

$$= \int \frac{e^{4x}}{\sqrt{3^2 - (e^{4x})^2}}\, dx$$

$$u = e^{4x}$$
$$du = 4e^{4x}\,dx$$
$$\frac{du}{4} = e^{4x}\,dx$$

$$= \frac{1}{4}\int \frac{du}{\sqrt{3^2 - u^2}}$$

$$= \frac{1}{4}\sin^{-1}\frac{u}{3} + C$$

$$= \frac{1}{4}\sin^{-1}\frac{e^{4x}}{3} + C$$

Ex: Six similar integrals

a) $\int \frac{1}{1+x^2}\,dx$

$= \arctan x + C$

b) $\int \frac{x}{1+x^2}\,dx$

$$u = 1 + x^2$$
$$du = 2x\,dx$$
$$\frac{du}{2} = x\,dx$$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+x^2| + C$

c) $\int \frac{x}{\sqrt{1+x^2}}\,dx$

$$\boxed{u = 1+x^2}$$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\left(2\,u^{1/2}\right) + C$

$= (1+x^2)^{1/2} + C$

d) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

e) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \\ -\tfrac{1}{2}\,du &= x\,dx \end{aligned}}$$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{\cancel{2}}\left(\cancel{2}\,u^{1/2}\right) + C$

$= -(1-x^2)^{1/2} + C$

f) $\int \frac{x}{1-x^2}\,dx$

$u = 1-x^2$

$= -\frac{1}{2}\int \frac{du}{u}$

$$= -\frac{1}{2} \ln|u| + C$$

$$= -\frac{1}{2} \ln|1-x^2| + C$$

<u>Ex</u>: Integrals that will come up often

$$\int e^x\,dx = e^x + C$$

$$\int e^{2x}\,dx = \frac{1}{2} e^{2x} + C$$

$$\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C \qquad (k \neq 0)$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos 4x\,dx = \frac{1}{4} \sin 4x + C$$

$$\int \cos kx\,dx = \frac{1}{k} \sin kx + C \quad (k \neq 0)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sin 9x\,dx = -\frac{1}{9} \cos 9x + C$$

$$\int \sin kx \, dx = -\frac{1}{k}\cos kx + C \quad (k \neq 0)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$
$$(a \neq 0)$$

8.2 Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

Comes from integrating

$$[uv]' = uv' + vu'$$

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Apply $\int dx$:

$$uv = \int u\frac{dv}{\cancel{dx}}\cancel{dx} + \int v\frac{du}{\cancel{dx}}\cancel{dx}$$

$$uv - \int v\,du = \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex: $\int x e^{2x}\,dx$

$$\boxed{\begin{array}{ll} u = x & dv = e^{2x}\,dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{array}}$$

Don't write $+C$

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

Ex: $\int x^3 \ln x\,dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^4}{4} \end{array}}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$