Indefinite Integral

$$\int x^2 dx = \frac{x^3}{3} + C$$

all possible antiderivatives of x^2

Definite Integral

$$\int_1^4 x^2 dx = \frac{x^3}{3}\Big|_1^4 = \frac{64}{3} - \frac{1}{3} = 21$$

area under the curve

4.4-4.5 Cont'd

Ex: Evaluate $\int_1^2 (x^2-4)\,dx$

$$= \left[\frac{x^3}{3} - 4x\right]_1^2$$

$$= \left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)$$

$$= -\frac{5}{3}$$

Ex: Evaluate $\int_0^2 2x(x^2+1)^3\,dx$

Method 1

$$\boxed{\begin{gathered} u = x^2+1 \\ \frac{du}{dx} = 2x \\ du = 2x\,dx \end{gathered}}$$

$$= \int_{x=0}^{x=2} u^3\,du$$

$$= \left.\frac{u^4}{4}\right|_{x=0}^{x=2}$$

$$= \left.\frac{1}{4}(x^2+1)^4\right|_0^2$$

$$= \frac{5^4}{4} - \frac{1}{4}$$

$$= \frac{624}{4}$$

$$= 156$$

Method 2

$$\boxed{\begin{gathered} u = x^2+1 \\ du = 2x\,dx \\ x=0 \Rightarrow u=1 \\ x=2 \Rightarrow u=5 \end{gathered}}$$

$$\text{Integral} = \int_1^5 u^3\,du$$

$$= \frac{u^4}{4}\Bigg|_1^5$$

$$= \frac{5^4}{4} - \frac{1}{4}$$

$$= 156$$

Ex: Evaluate $\displaystyle\int_0^1 \frac{x}{(x^2+1)^4}\,dx$

$$\boxed{\begin{aligned} u &= x^2+1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \\ x = 0 &\Rightarrow u = 1 \\ x = 1 &\Rightarrow u = 2 \end{aligned}}$$

$$= \frac{1}{2}\int_1^2 \frac{du}{u^4}$$

$$= \frac{1}{2}\int_1^2 u^{-4}\,du$$

$$= \frac{1}{2}\left(\frac{-1}{3}u^{-3}\right)\Big|_1^2$$

$$= \frac{-1}{6}\left(\frac{1}{8} - 1\right)$$

$$= \frac{7}{48}$$

5.2 and 5.4 Exponentials and Logs: Integration

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$$

Ex: $\displaystyle\int \frac{x}{x^2+4}\,dx$

$$\boxed{\begin{aligned} u &= x^2+4 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+4| + C$$

Ex: $\int \frac{1}{3x+7}\,dx$

$u = 3x+7$
$du = 3\,dx$
$\frac{du}{3} = dx$

$= \frac{1}{3}\int \frac{du}{u}$

$= \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|3x+7| + C$

Shortcut

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C \quad (a \neq 0)}$$

Ex: $\int \frac{3x^3 - 5x^2 + 10x - 3}{3x+1}\,dx$

Long Division

$f(x) = \frac{3x^3 - 5x^2 + 10x - 3}{3x+1}$

$$
\begin{array}{r}
x^2 - 2x + 4 \\
(3x+1)\overline{\big)\, 3x^3 - 5x^2 + 10x - 3} \\
-(3x^3 + x^2) \qquad\qquad \\
\hline
-6x^2 + 10x - 3 \\
-(-6x^2 - 2x) \qquad \\
\hline
12x - 3 \\
-(12x + 4) \\
\hline
-7
\end{array}
$$

$$f(x) = x^2 - 2x + 4 - \frac{7}{3x+1}$$

$$\text{Integral} = \int \left(x^2 - 2x + 4 - \frac{7}{3x+1}\right) dx$$

$$= \frac{x^3}{3} - x^2 + 4x - \frac{7}{3}\ln|3x+1| + C$$

<u>Ex</u>: $\int \cot\frac{\theta}{4}\, d\theta$

$$= \int \frac{\cos\frac{\theta}{4}}{\sin\frac{\theta}{4}}\, d\theta$$

$$u = \sin\frac{\theta}{4}$$

$$\frac{du}{d\theta} = \frac{1}{4}\cos\frac{\theta}{4}$$

$$du = \frac{1}{4}\cos\frac{\theta}{4}\,d\theta$$

$$4\,du = \cos\frac{\theta}{4}\,d\theta$$

$$= 4\int \frac{du}{u}$$

$$= 4\ln|u| + C$$

$$= 4\ln\left|\sin\frac{\theta}{4}\right| + C$$

Ex: $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{1}{4}(\ln x)^4 + C$$

Ex: $\int \frac{3x}{(x-2)^2}\,dx$

$u = x-2$

$\frac{du}{dx} = 1$

$du = dx$

$x = ?$

$x = u+2$

$= 3\int \frac{u+2}{u^2}\,du$

$= 3\int \left(\frac{1}{u} + \frac{2}{u^2}\right) du$

$= 3\left[\ln|u| - 2u^{-1}\right] + C$

$= 3\left[\ln|x-2| - 2(x-2)^{-1}\right] + C$

Ex: $\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx$

$u = \sqrt{x} - 1$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$$du = \frac{1}{2\sqrt{x}}\,dx$$
$$2\,du = \frac{dx}{\sqrt{x}}$$
$$\sqrt{x}\,dx = ?$$
$$2x\,du = \sqrt{x}\,dx$$
$$2(u+1)^2\,du = \sqrt{x}\,dx$$

$u = \sqrt{x} - 1$
$u + 1 = \sqrt{x}$
$(u+1)^2 = x$

$$= 2\int \frac{(u+1)^2\,du}{u}$$
$$= 2\int \frac{u^2+2u+1}{u}\,du$$
$$= 2\int \left(u + 2 + \frac{1}{u}\right) du$$
$$= 2\left[\frac{u^2}{2} + 2u + \ln|u|\right] + C$$
$$= 2\left[\frac{(\sqrt{x}-1)^2}{2} + 2(\sqrt{x}-1) + \ln|\sqrt{x}-1|\right] + C$$

$$\boxed{\int e^u\,du = e^u + C}$$

Ex: $\int e^{8x+4}\,dx$

$$\boxed{\begin{aligned} u &= 8x+4 \\ du &= 8\,dx \\ \frac{du}{8} &= dx \end{aligned}}$$

$$= \frac{1}{8}\int e^{u}\,du$$

$$= \frac{1}{8}e^{u} + C$$

$$= \frac{1}{8}e^{8x+4} + C$$

Shortcut

$$\boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C \quad (a \neq 0)}$$

Ex: $\int \frac{4e^{3x}}{2+5e^{3x}}\,dx$

$$\boxed{\begin{aligned} u &= 2+5e^{3x} \\ du &= 15e^{3x}\,dx \\ \frac{du}{15} &= e^{3x}\,dx \end{aligned}}$$

$$= \frac{4}{15}\int \frac{du}{u}$$

$$= \frac{4}{15} \ln|u| + C$$

$$= \frac{4}{15} \ln\left|2+5e^{3x}\right| + C$$

Ex: $\int e^{x}\sqrt{1+e^{x}}\,dx$

$$\boxed{\begin{aligned} u &= 1+e^{x} \\ du &= e^{x}\,dx \end{aligned}}$$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}\left(1+e^{x}\right)^{3/2} + C$$