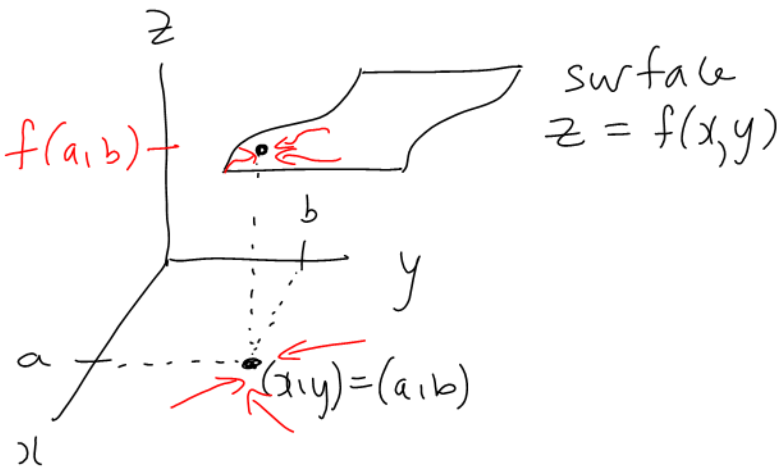
12.3 Limits and Continuity

Def

A function $f(x,y)$ is continuous at the point $(x,y)=(a,b)$ if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



Rephrased: A function $f(x,y)$ is continuous at a point if the surface has no hole or jump there.

Def

A function $f(x,y)$ is <u>continuous</u> if it is continuous at all points (x,y) in $\mathbb{R}^2$.

Typical continuous functions:

- Multivariable polynomials e.g. $x^4y^3 - 7x^3$
- Exponential, sine and cosine functions e.g. e^{2x-3y} or $\sin(x^2+y^2)$
- Sums, products or differences of the above

12.4 Partial Derivatives

Multivariable function $f(x,y)$

Notation

$\dfrac{\partial f}{\partial x}$ = partial derivative of f with respect to x

Calculation: Take derivative of f w.r.t. x (treat y as a constant)

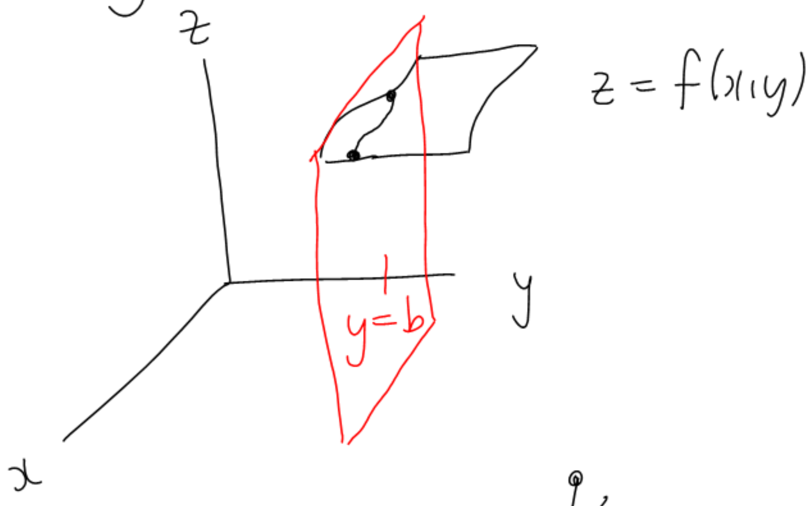
Ex: $f(x,y) = \sin(x^2+y^2) - 3e^{xy}$

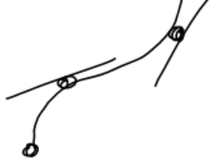
$$\frac{\partial f}{\partial x} = 2x\cos(x^2+y^2) - 3ye^{xy}$$

$$\frac{\partial f}{\partial y} = 2y\cos(x^2+y^2) - 3xe^{xy}$$

(Treat x as a constant)

Meaning of $\dfrac{\partial f}{\partial x}$





$\dfrac{\partial f}{\partial x}$ = slope of tangent line to the curve

Notation

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x = D_x[f(x,y)]$$

Ex: $z = x^3 \cos(2y) + e^{xy} \ln y$

Find z_x and z_y and $z_x \big|_{(x,y)=(2,\frac{\pi}{2})}$

$$z_x = 3x^2 \cos(2y) + ye^{xy} \ln y$$

$$z_y = -2x^3 \sin(2y) + \underbrace{\frac{e^{xy}}{y} + xe^{xy} \ln y}_{\text{Product Rule}}$$

$$z_x \Big|_{(2,\frac{\pi}{2})} = 12 \cos \pi + \frac{\pi}{2} e^{\pi} \ln\left(\frac{\pi}{2}\right)$$

$$= -12 + \frac{\pi}{2} e^{\pi} \ln \frac{\pi}{2}$$

Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate w.r.t. x then y

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Ex: Find all 2^{nd}-order partial derivatives of $f = x^4 + x^2y^2 + y^3$

$f_x = 4x^3 + 2xy^2$ $\qquad$ $f_y = 2x^2y + 3y^2$

$f_{xx} = 12x^2 + 2y^2$ $\quad$ $f_{xy} = 4xy$ $\quad$ $f_{yx} = 4xy$ $\quad$ $f_{yy} = 2x^2 + 6y$

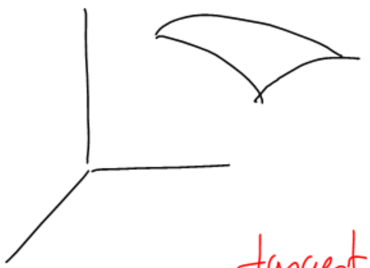
OR

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2y^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 4xy \quad \frac{\partial^2 f}{\partial x \partial y} = 4xy \quad \frac{\partial^2 f}{\partial y^2} = 2x^2 + 6y$$

FACT

$f_{xy} = f_{yx}$ if f_{xy} and f_{yx} are both continuous.

Surface $z = f(x,y)$



Tangent plane to a surface at a given point.

FACT

A normal vector to the tangent plane

is $\vec{n} = [-z_x, -z_y, 1]$.

<u>Ex</u>: Find the equation of the plane tangent to $z = 12 - 4x^2 - 3y^2$ at the point $(x, y, z) = (1, 2, -4)$.

$$z_x = -8x = -8$$

$$z_y = -6y = -12$$

$$\vec{n} = [-z_x, -z_y, 1]$$
$$= [8, 12, 1]$$

Normal form for a plane

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

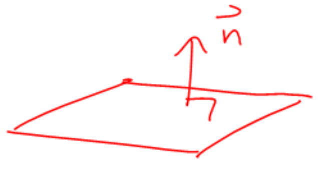
$$\begin{bmatrix} 8 \\ 12 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$8x + 12y + z = 28$$

More about Tangent Planes

Recall $\vec{n} = [-z_x, -z_y, 1]$

Q: Where is a tangent plane horizontal?

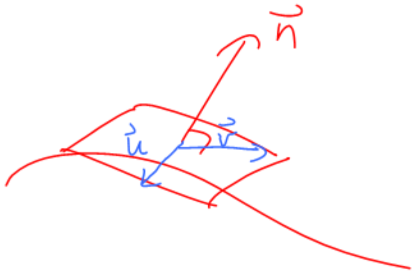


Horizontal Tangent Plane

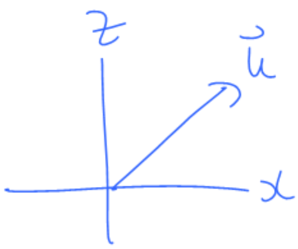
$\Rightarrow \quad \vec{n} = [0, 0, 1]$

$\Rightarrow \quad z_x = 0$ and $z_y = 0$

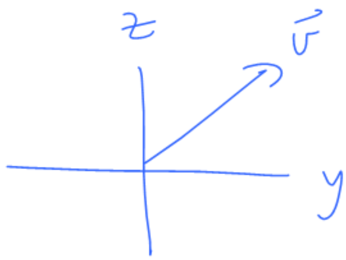
Q: why does $\vec{n} = [-z_x, -z_y, 1]$?



$\vec{u}, \vec{v}$: direction vectors for the tangent plane parallel to x and y-axis



$\vec{u} = [?, ?, ?]$



$\vec{v} = [?, ?, ?]$

$$\vec{n} = \vec{u} \times \vec{v}$$

To Be Continued