Exam Review

Problems are on website.

(16)

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \\ x=0 &\Rightarrow u=1 \\ x\to\infty &\Rightarrow u\to\infty \end{aligned}}$$

$$\text{Integral} = \int_1^{\infty} \frac{du}{1+u^2}$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{du}{1+u^2}$$

$$= \lim_{b\to\infty} \arctan u \Big|_1^b$$

$$= \lim_{b\to\infty} \arctan b - \arctan 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

17 a) $a_1 = e \quad a_2 = \frac{e^2}{2} \quad a_3 = \frac{e^3}{3}$

$$\lim_{n\to\infty} \frac{e^n}{n} \leftarrow \left(\frac{\infty}{\infty}\right)$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{e^n}{1}$$

$$= \infty$$

b) $a_0 = 0 \quad a_1 = \frac{4}{\sqrt{2}} \quad a_2 = \frac{8}{\sqrt{5}}$

$$\lim_{n\to\infty} \frac{4n}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} 4\sqrt{\frac{n^2}{n^2+1}}$$

$$= \lim_{n\to\infty} 4\sqrt{\frac{1}{1+\left(\frac{1}{n^2}\right)}}$$

$$= 4$$

(18) a) Partial Fractions

$$\frac{7}{(n+3)(n+4)} = \frac{A}{n+3} + \frac{B}{n+4}$$

$$7 = A(n+4) + B(n+3)$$

Sub $n=-4$: $\quad 7 = B(-1) \Rightarrow B = -7$

$n=-3$: $\quad 7 = A$

$$\text{Series} = \sum_{n=2}^{\infty} \left(\frac{7}{n+3} - \frac{7}{n+4} \right)$$

$$= \left(\frac{7}{5} \cancel{- \frac{7}{6}} \right) + \left(\frac{7}{6} \cancel{- \frac{7}{7}} \right) + \dots$$

$$= \frac{7}{5} - \lim_{n\to\infty} \frac{7}{n+4}$$

$$= \frac{7}{5}$$

b) $$\sum_{n=2}^{\infty} \frac{2^{n+1}}{7^n} = \frac{2^3}{7^2} + \frac{2^4}{7^3} + \dots$$

$\times r$

$$\boxed{\begin{aligned} a &= \frac{2^3}{7^2} = \frac{8}{49} \\ r &= \frac{2}{7} \end{aligned}}$$

$$\text{Series} = \frac{a}{1-r}$$

$$= \frac{\left(\frac{8}{49}\right)}{\left(\frac{5}{7}\right)}$$

$$= \frac{8}{49} \cdot \frac{7}{5}$$

$$= \frac{8}{35}$$

(19) Compute $\displaystyle\int_N^\infty \frac{2}{x^2}\,dx$

Then $\displaystyle\int_N^\infty \frac{2}{x^2}\,dx \le 0.1$

$$\int_N^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty} \int_N^b 2x^{-2}\,dx$$

$$= \lim_{b\to\infty} -2x^{-1}\Big|_N^b$$

$$= \lim_{b\to\infty} \frac{-2}{b} + \frac{2}{N}$$

$$= \frac{2}{N}$$

$$\frac{2}{N} \leq 0.1$$

$$2 \leq 0.1N$$

$$20 \leq N$$

20

a) $S_3 = -1 + \frac{1}{4} - \frac{1}{9}$

$$= \frac{-31}{36}$$

b) $|R_3| \leq a_4 \leftarrow$ unsigned term

$\leq \frac{1}{16}$

c) $\frac{-31}{36} - \frac{1}{16} \leq \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \leq \frac{-31}{36} + \frac{1}{16}$

21 a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$

p-series with $p = 1.1$

Converges

b) $\sum_{n=1}^{\infty} n^2 e^{(-n^3)}$

$f(x) = x^2 e^{-x^3}$

$f(x)$ is continuous ✓

$f(x)$ is positive for $x \geq 1$ ✓

$$f'(x) = x^2\left(-3x^2 e^{-x^3}\right) + 2x e^{-x^3}$$

$$= \left(2x - 3x^4\right) e^{-x^3}$$

$f'(x) < 0$ for $x \geq 1$ ✓

$$\int_1^\infty x^2 e^{-x^3}\,dx$$

$$\boxed{\begin{aligned} u &= -x^3 \\ du &= -3x^2\,dx \\ -\tfrac{1}{3}\,du &= x^2\,dx \\ x = 1 &\Rightarrow u = -1 \\ x \to \infty &\Rightarrow u \to -\infty \end{aligned}}$$

$$= -\frac{1}{3}\int_{-1}^{-\infty} e^u\,du$$

$$= \lim_{b \to -\infty} -\frac{1}{3}\int_{-1}^{b} e^u\,du$$

$$= \lim_{b \to -\infty} -\frac{1}{3} e^u \Big|_{-1}^{b}$$

$$= \lim_{b \to -\infty} -\frac{1}{3}e^{b} + \frac{1}{3}e^{-1}$$

$$= \frac{1}{3}e^{-1}$$

Series converges.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

n^{th} Term Test

$$\lim_{n \to \infty} \frac{n}{2n+1} \overset{(H)}{=} \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$$

Series diverges.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$

Alternating Series ✓

$a_n = \frac{1}{2n}$

$$\frac{1}{2(n+1)} \leq \frac{1}{2n}$$ ✓

$$\lim_{n \to \infty} a_n = 0$$ ✓

Series converges.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{(n^3)}}$

Direct Comparison

$$0 < \frac{1}{3^{(n^3)}} \leq \frac{1}{3^n} \text{ for } n \geq 1$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric)

Conclude the series converges.

f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{7n^2+2}$

Limit Comparison.

Dominant term $= \dfrac{\sqrt{n}}{n^2} = n^{-3/2}$

Let $a_n = \frac{\sqrt{n}+3}{7n^2+2}$ and $b_n = n^{-3/2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}+3}{7n^2+2}\left(n^{3/2}\right)$$

$$= \lim_{n\to\infty} \frac{n^2+3n^{3/2}}{7n^2+2}$$

$$= \lim_{n\to\infty} \frac{1+3n^{-1/2}}{7+2n^{-2}}$$

$$= \frac{1}{7}$$

$$0 < L < \infty \checkmark$$

$\sum_{n=1}^{\infty} n^{-3/2}$ Converges (p-series)

Conclude $\sum_{n=1}^{\infty} a_n$ Converges.

g) Ratio Test h) Root Test