12.4 Tangent and Normal Vectors Cont'd

Tangential Component of acceleration

$$a_T = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$$

Normal component of acceleration

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$$

Ex: Given $\vec{r}(t) = [t, \frac{t^2}{2}, \frac{t^3}{3}]$.
Find a_T and a_N.

$$\vec{v}(t) = [1, t, t^2]$$

$$\vec{a}(t) = [0, 1, 2t]$$

$$\vec{v} \cdot \vec{a} = 0 + t + 2t^3$$
$$= t + 2t^3$$

$$\|\vec{v}\| = \sqrt{1^2 + t^2 + (t^2)^2}$$

$$= \sqrt{t^4 + t^2 + 1}$$

$$a_T = \frac{t + 2t^3}{\sqrt{t^4 + t^2 + 1}}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & t & t^2 \\ 0 & 1 & 2t \end{vmatrix}$$

$$= \vec{i}\,(2t^2 - t^2) - \vec{j}\,(\;2t\;) + \vec{k}\,(\;1\;)$$

$$= [\;t^2\;, -2t\;, \;1\;]$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{(t^2)^2 + (-2t)^2 + 1^2}$$

$$= \sqrt{t^4 + 4t^2 + 1}$$

$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$$

$$= \frac{\sqrt{t^4 + 4t^2 + 1}}{\sqrt{t^4 + t^2 + 1}}$$

<u>Ex:</u> Let $\vec{r} = [e^t, e^t\cos t, e^t \sin t]$

Find $a_T(t)$ and $a_N(t)$ evaluated at $t = 0$.

$$\vec{v} = [e^t, -e^t \sin t + e^t \cos t\;, \;e^t \cos t + e^t \sin t]$$

$$\vec{a} = [e^t, \cancel{-e^t\cos t} \underbrace{- e^t \sin t \; - e^t \sin t}_{-2e^t \sin t} + \cancel{e^t \cos t}\;,$$

$$\cancel{-e^t \sin t} + \underbrace{e^t \cos t + e^t \cos t}_{2e^t \cos t} + \cancel{e^t \sin t}\;]$$

$\vec{v}(0) = [1, 1, 1]$

$\vec{a}(0) = [1, 0, 2]$

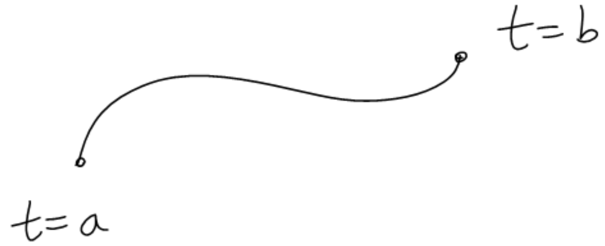
$$a_T(0) = \frac{\vec{v}(0) \cdot \vec{a}(0)}{\|\vec{v}(0)\|} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$a_N(0) = \frac{\|\vec{v}(0) \times \vec{a}(0)\|}{\|\vec{v}(0)\|}$$

$$\boxed{\begin{aligned} &\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\ &= \vec{i}(2) - \vec{j}(1) + \vec{k}(-1) \\ &= [2, -1, -1] \end{aligned}}$$

$$= \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

12.5 Arc Length



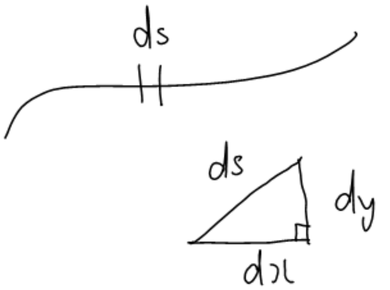
$s =$ arc length

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (2D)$$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad (3D)$$

or $$S = \int_a^b \| \vec{v}(t) \| \, dt \qquad (2D \text{ or } 3D)$$

Why?



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) dt^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \| \vec{v}(t) \| \, dt$$

<u>Ex</u>: Find the arc length of

$\vec{r}(t) = [\cos 3t, \sin 3t, 2t]$ on $0 \le t \le 4\pi$.

$$\vec{v}(t) = [-3\sin 3t,\ 3\cos 3t,\ 2]$$

$$\|\vec{v}(t)\| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 4}$$

$$= \sqrt{9\underbrace{(\sin^2 3t + \cos^2 3t)}_{1} + 4}$$

$$= \sqrt{13}$$

$$s = \int_a^b \|\vec{v}(t)\|\,dt$$

$$= \int_0^{4\pi} \sqrt{13}\,dt$$

$$= \sqrt{13}\,t \Big|_0^{4\pi}$$

$$= \sqrt{13}\,(4\pi)$$

$$= 4\pi\sqrt{13}$$

Ex: Find the arc length of

$\vec{r}(t) = [2t^2+1,\ t^3+5]$ on $0 \le t \le 1$.

$$\vec{v}(t) = [4t,\ 3t^2]$$

$$\|\vec{v}(t)\| = \sqrt{(4t)^2 + (3t^2)^2}$$

$$= \sqrt{16t^2 + 9t^4}$$

$$= \sqrt{t^2(16+9t^2)}$$

$$= |t|\sqrt{16+9t^2}$$

$$= t\sqrt{16+9t^2} \qquad (0 \le t \le 1)$$

$$S = \int_a^b \|\bar{v}(t)\|\,dt$$

$$= \int_0^1 t\sqrt{16+9t^2}\,dt$$

To Be Cont'd