FINAL EXAM

Thurs Dec 14
1:30–4:30pm
TEC 175

No Music Allowed
Bring: calculator, earplugs

14 Questions

Sections	% of Marks on Exam
8.2–8.5, 5.6, 8.8	30
9.1–9.10	28
10.2–10.5	25
12.1–12.5	17

12.3 Velocity and Acceleration Cont'd

Projectile Motion in 2D

Position (in m) after t seconds:

$$\vec{r} = \left[(v_0 \cos\theta)t,\ y_0 + (v_0 \sin\theta)t - \frac{gt^2}{2}\right]$$

where:

v_0 = initial speed (m/s)

θ = angle of inclination (measured to horizontal)

y_0 = initial height (m)

$g = 9.8\ m/s^2$ (or $g = 32\ ft/s^2$)

Ex: A projectile has
$v_0 = 10$ m/s, $\theta = 30°$, $y_0 = 2$ m.

a) Find $\vec{r}$

$$\vec{r} = [(10\cos 30°)t,\ 2 + (10\sin 30°)t - 4.9t^2]$$
$$= [5\sqrt{3}\,t,\ 2 + 5t - 4.9t^2]$$

b) Find $\vec{v}$

$$\vec{v} = [5\sqrt{3},\ 5 - 9.8t]$$

c) Find the maximum height.

Occurs when (y-component of $\vec{v}$) = 0

$$5 - 9.8t = 0$$
$$t = \frac{5}{9.8}$$

$$\text{Maximum height} = (\text{y-component of } \vec{r})\Big|_{t = \frac{5}{9.8}}$$
$$= 2 + 5\left(\frac{5}{9.8}\right) - 4.9\left(\frac{5}{9.8}\right)^2$$
$$\approx 3.3 \text{ m}$$

d) Find the speed when it hits the ground.

Occurs when (y-component of $\vec{r}$) = 0

$$2 + 5t - 4.9t^2 = 0$$

$$-4.9t^2 + 5t + 2 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4(-4.9)(2)}}{-9.8}$$

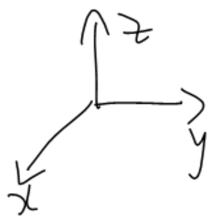
$$\approx 1.328, \ \sout{-0.307}$$

$$\|\vec{v}(t)\| = \sqrt{(5\sqrt{3})^2 + (5 - 9.8t)^2}$$

$$\|\vec{v}(1.328)\| \approx 12 \text{ m/s}$$

Ex: A ball is thrown with spin such that

$$\vec{a} = [0.60, 0, -9.8],$$
$$\vec{v}(0) = [0, 10, 10],$$
$$\vec{r}(0) = [0, 0, 0].$$



Find where, and at what speed, the ball hits the ground.

$$\vec{v} = \int \vec{a}\, dt$$

$$= [0.60t, 0, -9.8t] + \vec{c}_1$$

Sub $t=0$: $[0, 10, 10] = \vec{0} + \vec{c}_1$

$$\vec{c}_1 = [0, 10, 10]$$

$$\boxed{\vec{v} = [0.60t,\ 10,\ 10 - 9.8t]}$$

$$\vec{r} = \int \vec{v}\, dt$$

$$= [0.30t^2,\ 10t,\ 10t - 4.9t^2] + \vec{C}_2$$

Sub $t=0$: $\vec{0} = \vec{0} + \vec{C}_2$

$$\vec{C}_2 = \vec{0}$$

$$\boxed{\vec{r} = [0.30t^2,\ 10t,\ 10t - 4.9t^2]}$$

Ball hits the ground when (z-component of $\vec{r}$) = 0

$$10t - 4.9t^2 = 0$$

$$t(10 - 4.9t) = 0$$

$$~~t = 0~~,\quad t = \frac{10}{4.9}$$

Position at landing $= \vec{r}\left(\frac{10}{4.9}\right)$

$$= [1.2,\ 20,\ 0]$$

$$\|\vec{v}(t)\| = \sqrt{(0.60t)^2 + 10^2 + (10 - 9.8t)^2}$$

Speed at landing $= \left\|\vec{v}\left(\frac{10}{4.9}\right)\right\|$

$$\approx 14 \text{ m/s}$$

12.4 Tangent and Normal Vectors

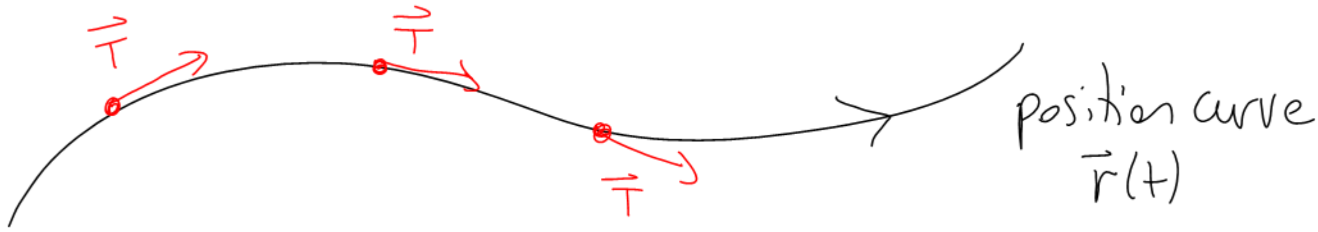
A curve $\vec{r}(t)$ is <u>smooth</u> on $a \le t \le b$ if $\vec{r}\,'(t)$ is continuous and nonzero on $a \le t \le b$.

Def

A <u>unit tangent vector</u> to a smooth curve $\vec{r}(t)$ is :

$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$$

It has length 1 and it's tangent to $\vec{r}(t)$.



<u>Ex</u>: Show that $\vec{T}\,'(t) \perp \vec{T}(t)$ for all t.

$$\|\vec{T}(t)\| = 1$$

$$\|\vec{T}(t)\|^2 = 1$$

$$\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$$

$$\vec{T}(t) \cdot \vec{T}(t) = 1$$

Take $\frac{d}{dt}$: $\vec{T}(t)\cdot\vec{T}'(t) + \vec{T}'(t)\cdot\vec{T}(t) = 0$

$$2\left[\vec{T}'(t)\cdot\vec{T}(t)\right] = 0$$

$$\vec{T}'(t)\cdot\vec{T}(t) = 0$$

$$\vec{T}'(t) \perp \vec{T}(t) \quad \text{for all } t$$

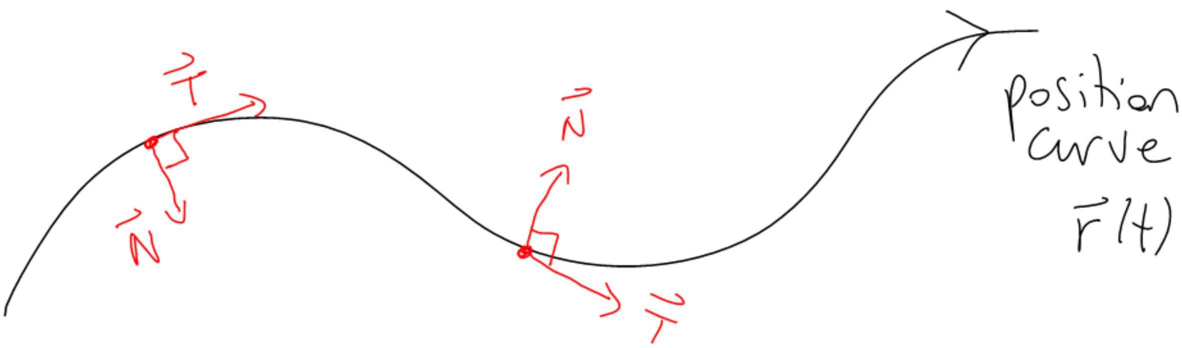
Def

A <u>unit normal vector</u> to a smooth curve $\vec{r}(t)$ is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

It's perpendicular to $\vec{T}(t)$ (and $\vec{r}(t)$), pointing towards the concave side of $\vec{r}(t)$.

It has length 1.



FACT

$$\vec{a}(t) = \underbrace{a_T(t)}\,\vec{T}(t) + \underbrace{a_N(t)}\,\vec{N}(t)$$

↖ functions of t ↗

where $a_T(t) = \dfrac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}$ "tangential component of acceleration"

$a_N(t) = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$ "normal component of acceleration"