10.5 Cont'd

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

why?

$$A_{circle} = \pi r^2$$

$$A_{sector} = \pi r^2 \left(\frac{\theta}{2\pi}\right)$$
$$= \frac{1}{2} r^2 \theta$$

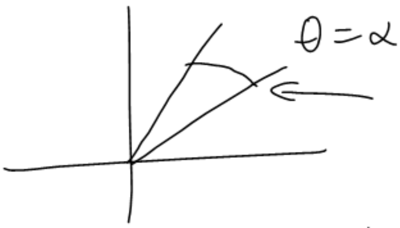
$$dA = \frac{1}{2} r^2 \, d\theta$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$\theta = \beta$

$\theta = \alpha$



$r = f(\theta)$

Arc Length $\quad s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

why?

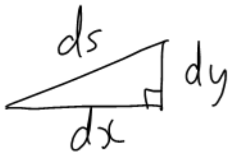
$x = r\cos\theta$

$\frac{dx}{d\theta} = r(-\sin\theta) + (\cos\theta)\frac{dr}{d\theta}$

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + (\sin\theta)\frac{dr}{d\theta}$

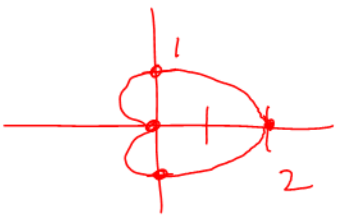
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2\sin^2\theta \cancel{- 2\sin\theta\cos\theta\, r\frac{dr}{d\theta}} + \cos^2\theta\left(\frac{dr}{d\theta}\right)^2$$
$$+ r^2\cos^2\theta \cancel{+ 2\sin\theta\cos\theta\, r\frac{dr}{d\theta}} + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2$$
$$= r^2\left(\sin^2\theta + \cos^2\theta\right) + \left(\frac{dr}{d\theta}\right)^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r^2 + \left(\frac{dr}{d\theta}\right)^2$$



$$ds = \sqrt{(dx)^2 + (dy)^2}$$
$$= \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right](d\theta)^2}$$
$$= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$
$$= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex: Find the arc length of $r = 1 + \cos\theta$

$$\frac{dr}{d\theta} = -\sin\theta$$



Curve is traced out over $0 \le \theta \le 2\pi$

By symmetry,

$s =$ double the arc length over $0 \leq \theta \leq \pi$

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_{0}^{\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$= 2\int_{0}^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 2\int_{0}^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$\boxed{\begin{aligned} \frac{1+\cos 2\alpha}{2} &= \cos^2\alpha \\ 2 + 2\cos 2\alpha &= 4\cos^2\alpha \\ 2 + 2\cos\theta &= 4\cos^2\frac{\theta}{2} \end{aligned}}$$

$$= 2\int_{0}^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 2\int_{0}^{\pi} \left|2\cos\frac{\theta}{2}\right| d\theta$$

$$= 2\int_{0}^{\pi} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4\left[2\sin\frac{\theta}{2}\right]_{0}^{\pi}$$

$= 8(1-0)$
$= 8$

12.1 Vector-Valued Functions

Recall: parametric curve (Section 10.2)

$$\begin{cases} x = x(t) \\ y = y(t) \\ a \le t \le b \end{cases}$$

Can be described with the position vector

$\vec{r}(t) = [x(t), y(t)]$ $(a \le t \le b)$

or $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ $(a \le t \le b)$

Ex: $\vec{r}(t) = [t, t^2]$

Find $\vec{r}(6)$

$\vec{r}(6) = [6, 36]$ position at $t=6$

Note: $\vec{r}(t)$ is called a vector-valued function.

Ex: Find the position vector.

$$\begin{cases} x = 4+2t \\ y = 1+5t \\ z = 3+t \\ 0 \le t \le 1 \end{cases}$$

$\vec{r}(t) = [4+2t, 1+5t, 3+t]$ ✓

or $\vec{r}(t) = (4+2t)\,\vec{i} + (1+5t)\,\vec{j} + (3+t)\,\vec{k}$ ✓

$(0 \le t \le 1)$

Ex: Find the position vector of the line segment from $(1,2,3)$ to $(-4,6,8)$.

x: starts at 1, net change of -5
y: 2, $+4$
z: 3, $+5$

$$\begin{cases} x = 1-5t \\ y = 2+4t \\ z = 3+5t \end{cases} \qquad (0 \le t \le 1)$$

$$\vec{r}(t) = [1-5t,\ 2+4t,\ 3+5t] \qquad (0 \le t \le 1)$$

12.2 Derivatives and Integrals of Vector-Valued Functions

FACT

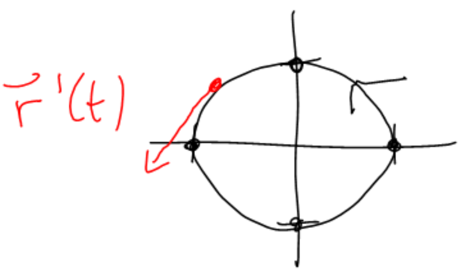
If $\vec{r}(t) = [x(t), y(t), z(t)]$

then $\vec{r}\,'(t) = [x'(t), y'(t), z'(t)]$

Note: $\vec{r}\,'(t)$ is a tangent vector to the position curve.

Ex: Find $\vec{r}\,'(t)$ for $\vec{r}(t) = [\cos t, \sin t]$

$\vec{r}\,'(t) = [-\sin t, \cos t]$



6 Properties

Let: c be a constant

$f(t)$ be a function of t

$\vec{r}(t), \vec{s}(t)$ be vector-valued functions

1) $[c\,\vec{r}(t)]' = c\,\vec{r}\,'(t)$

2) $[\vec{r}(t) \pm \vec{s}(t)]' = \vec{r}\,'(t) \pm \vec{s}\,'(t)$

3) $[f(t)\,\vec{r}(t)]' = f(t)\,\vec{r}\,'(t) + f'(t)\,\vec{r}(t)$

Product Rule

4) $[\vec{r}(t) \cdot \vec{s}(t)]' = \vec{r}(t) \cdot \vec{s}\,'(t) + \vec{r}\,'(t) \cdot \vec{s}(t)$

Product Rule

5) $[\vec{r}(t) \times \vec{s}(t)]' = \vec{r}(t) \times \vec{s}\,'(t) + \vec{r}\,'(t) \times \vec{s}(t)$

Product Rule

6) $[\vec{r}(f(t))]' = \vec{r}\,'(f(t))\,f'(t)$

Chain Rule

Ex: Find $\frac{d}{dt}\, t^2[9t, t^3]$ two different ways.

$$\frac{d}{dt}\, t^2[9t, t^3]$$

$$= \frac{d}{dt}\,[9t^3, t^5]$$

$$= [27t^2, 5t^4]$$

$$\frac{d}{dt} t^2 [9t, t^3]$$

$$= t^2 [9, 3t^2] + 2t [9t, t^3]$$

$$= [9t^2, 3t^4] + [18t^2, 2t^4]$$

$$= [27t^2, 5t^4] \checkmark$$

Ex: Let $\vec{r}(t) = [t^2+1, 7t]$

Find $\frac{d}{dt} \vec{r}(2t)$ two different ways.

$$\frac{d}{dt} \vec{r}(2t) = \frac{d}{dt} [4t^2+1, 14t]$$

$$= [8t, 14]$$

$$\frac{d}{dt} \vec{r}(2t) = \vec{r}\,'(2t)\ (2)$$

$$\boxed{\vec{r}\,'(t) = [2t, 7]}$$

$$= [4t, 7]\ (2)$$

$$= [8t, 14] \checkmark$$