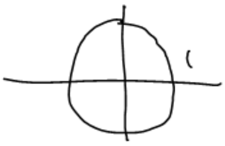
10.4 Polar Curves Cont'd

Review: Solving Trig Equations

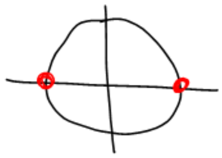
Method 1: Unit Circle



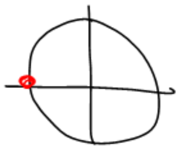
$\cos\theta$ is the x-coordinate of a point on the unit circle

$\sin\theta$ is the y-coordinate "

$\sin\theta = 0 \implies \theta = 0, \pi$



$\cos\theta = -1 \implies \theta = \pi$



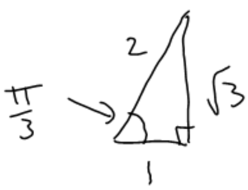
Use this method to solve $\sin\theta = 0, \pm 1$

$\cos\theta = 0, \pm 1$

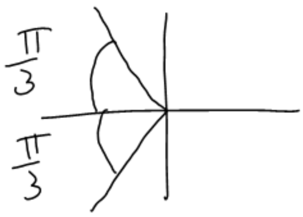
Method 2: Reference Angle

Solve $\cos\theta = -\frac{1}{2}$ on $[0, 2\pi)$

$\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ is the <u>reference angle</u>.

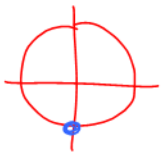


The reference angle is always between 0 and $\frac{\pi}{2}$.
The reference angle is measured to the x-axis.



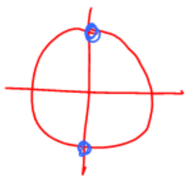
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Ex: Solve $\sin\theta = -1$ on $[0, 2\pi)$



$\theta = \frac{3\pi}{2}$

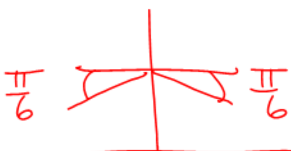
Ex: Solve $\cos\theta = 0$ on $[0, 2\pi)$



$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Ex: Solve $\sin\theta = -\frac{1}{2}$ on $[0, 2\pi)$

reference angle $= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$



$\boxed{\theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$

FACT

$r = f(\theta)$ as a parametric curve:

$$\begin{cases} x = r\cos\theta \\ \quad = f(\theta)\cos\theta \\ y = r\sin\theta \\ \quad = f(\theta)\sin\theta \end{cases}$$

It has $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$

Recall: Vertical tangent if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$

Horizontal " if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

Ex: Given $r = 1 + \cos\theta$

For which θ is the tangent vertical or horizontal?

$$x = r\cos\theta$$
$$= (1+\cos\theta)\cos\theta$$
$$= \cos\theta + \cos^2\theta$$

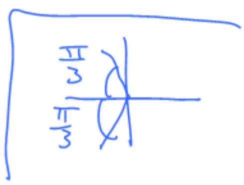
$$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta(-\sin\theta)$$

Set $\frac{dx}{d\theta} = 0$: $\quad 0 = -\sin\theta(1 + 2\cos\theta)$

$\sin\theta = 0 \qquad 1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$



$$y = r\sin\theta$$
$$= (1+\cos\theta)\sin\theta$$

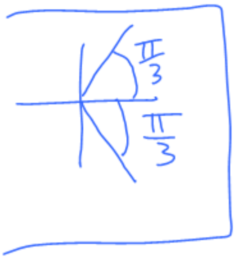
$$\frac{dy}{d\theta} = (1+\cos\theta)\cos\theta + \sin\theta(-\sin\theta) \qquad \text{Product Rule}$$

$$= \cos\theta + \cos^2\theta \underbrace{- \sin^2\theta}_{-(1-\cos^2\theta)}$$

$$= \cos\theta + 2\cos^2\theta - 1$$
$$= 2\cos^2\theta + \cos\theta - 1$$

$\frac{dy}{d\theta} = 0$:
$$0 = 2\cos^2\theta + \cos\theta - 1$$
$$0 = (2\cos\theta - 1)(\cos\theta + 1)$$

$$2\cos\theta - 1 = 0 \qquad \cos\theta + 1 = 0$$
$$\cos\theta = \frac{1}{2} \qquad \cos\theta = -1$$



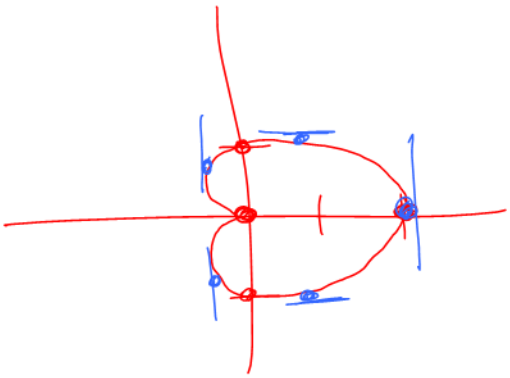
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$$

Vertical Tangent: $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

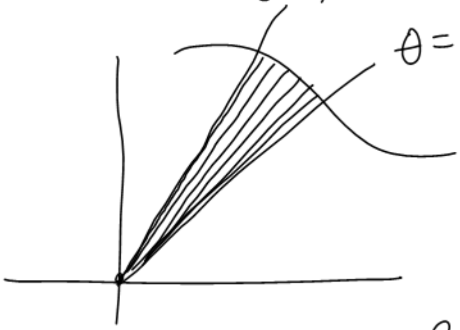
Horizontal ": $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$r = 1 + \cos\theta$$

θ	r
0	2
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	1



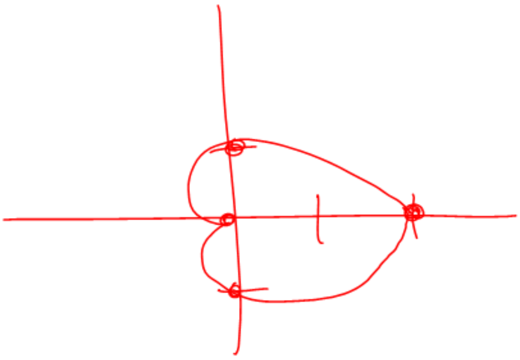
10.5 Area and Arc Length in Polar



Area $\quad A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$

Ex: Find the area inside $r = 1 + \cos\theta$

θ	r
0	2
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	1



Curve is traced out over $0 \le \theta < 2\pi$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} [1 + \cos\theta]^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi} (1 + 2\cos\theta + \underbrace{\cos^2\theta}_{\frac{1}{2} + \frac{\cos 2\theta}{2}}) \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[3\pi - 0\right]$$

$$= \frac{3\pi}{2}$$

ASIDE

$$\int \sin\theta\, d\theta = -\cos\theta + C$$

$$\int \cos\theta\, d\theta = \sin\theta + C$$

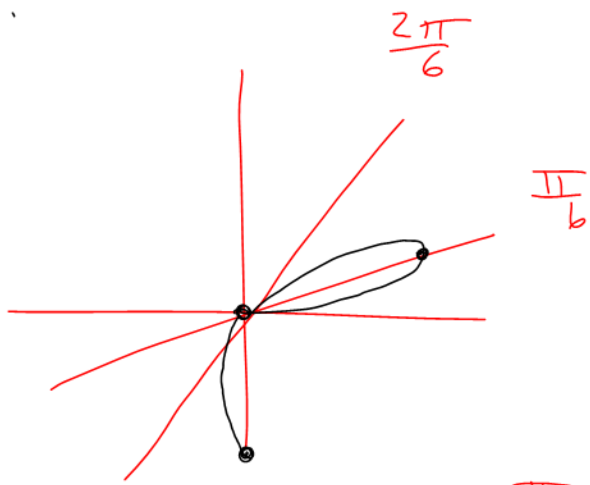
$$\int \sin 2\theta\, d\theta = -\frac{\cos 2\theta}{2} + C$$

$$\int \cos 2\theta\, d\theta = \frac{\sin 2\theta}{2} + C$$

$$\int \frac{\sin 2\theta}{3}\, d\theta = -\frac{\cos 2\theta}{6} + C$$

Ex: Find the area of one petal of $r = \sin 3\theta$.

θ	r
0	0
$\frac{\pi}{6}$	1
$\frac{2\pi}{6}$	0
$\frac{3\pi}{6}$	-1



Let $0 \le \theta \le \frac{\pi}{3}$

OR

Solve $r=0$ to find where petals start.

$$\sin 3\theta = 0$$

$$3\theta = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$$

$$0 \le \theta \le \frac{\pi}{3}$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/3} \sin^2 3\theta \, d\theta$$

To Be Continued