Review

(1) Decide whether the series converges or diverges. Show your work.

$$\sum_{n=1}^{\infty} \frac{7\sqrt{n}+8}{8n+7\sqrt{n}}$$

Limit Comparison Test

Dominant terms $= \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

Compare it with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$\frac{7\sqrt{n}+8}{8n+7\sqrt{n}} > 0$ and $\frac{1}{\sqrt{n}} > 0$ for $n \geq 1$ ✓

$$L = \lim_{n\to\infty} \frac{\left(\frac{7\sqrt{n}+8}{8n+7\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)}$$

$$= \lim_{n\to\infty} \left(\frac{7\sqrt{n}+8}{8n+7\sqrt{n}}\right)\left(\frac{\sqrt{n}}{1}\right)$$

$$= \lim_{n\to\infty} \frac{7n+8\sqrt{n}}{8n+7\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{7+\frac{8}{\sqrt{n}}}{8+\frac{7}{\sqrt{n}}}$$

$$= \frac{7}{8}$$

$$0 < L < \infty$$ ✓

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} \frac{7\sqrt{n}+8}{8n+7\sqrt{n}}$ diverges

2 a) Find the 3^{rd} degree Taylor polynomial for $f(x) = \ln x$ centred at $c=1$.

$$f(x) = \ln x \qquad f(1) = 0$$

$$f'(x) = x^{-1} \qquad f'(1) = 1$$

$$f''(x) = -x^{-2} \qquad f''(1) = -1$$

$$f'''(x) = 2x^{-3} \qquad f'''(1) = 2$$

$$P_3(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \dots$$

$$= \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$ ✓

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$ ✓

b) Find an upper bound for the error

$$|R_3(0.8)|$$

$$|R_3(x)| = \left|\frac{f^{(4)}(z)}{4!}(x-c)^4\right|$$

$$f^{(4)}(x) = -6x^{-4}$$

$$f^{(4)}(z) = -6z^{-4}$$

$$|R_3(x)| = \left|\frac{6z^{-4}}{4!}(x-c)^4\right|$$

where z is between $\cancel{x}$ and $\cancel{c}$
0.8 1

$$\leq \frac{6(0.8)^{-4}}{4!}(0.2)^4$$

$$\leq 0.000977$$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ converges by the Alternating Series Test.

Find the smallest N so that

$$|R_N| \leq 0.01$$

$$a_n = \frac{1}{n^2+1}$$

$$a_{N+1} = \frac{1}{(N+1)^2+1}$$

Let $a_{N+1} \le 0.01$
(Guarantees $|R_N| \le 0.01$)

$$\frac{1}{(N+1)^2+1} \le 0.01$$

$$\cancel{\frac{1}{0.01}}\ 100 \le (N+1)^2+1$$

$$99 \le (N+1)^2$$

$$\sqrt{99} \le N+1$$

$$\sqrt{99} - 1 \le N$$

$$N \ge 8.95$$

$$\boxed{N=9}$$

4 $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges by the Integral Test.

a) Find an upper bound for R_5.

$$R_5 \le \int_5^{\infty} \frac{1}{x^3}\,dx$$

$$\le \lim_{b\to\infty} \int_5^{b} x^{-3}\,dx$$

$$\le \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]\Big|_5^b$$

$$\le \lim_{b\to\infty} -\frac{1}{2}b^{-2} + \frac{1}{2}5^{-2}$$

$$\le 0.02$$

b) Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ given $S_5 \approx 1.1857$

$$S_5 \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le S_5 + 0.02$$

5. For which series can we evaluate the sum?

Geometric or Telescoping

$$\sum_{n=2}^{\infty} \frac{13}{7}(0.89)^{n-1}$$

$$= \frac{a}{1-r}$$

$$= \frac{\frac{13}{7}(0.89)^1}{0.11}$$

$$\sum_{n=7}^{\infty} \frac{8}{(n+4)(n+5)}$$

$\vdots$

$$= \sum_{n=7}^{\infty} \left[\frac{8}{n+4} - \frac{8}{n+5}\right]$$

$$= \frac{8}{11} - \cancel{\frac{8}{12}} + \cancel{\frac{8}{12}} - \frac{8}{13} + \ldots$$

≈ 15.03

$$= \frac{8}{11} - \lim_{n\to\infty} \frac{8}{n+5}$$

$$= \frac{8}{11}$$