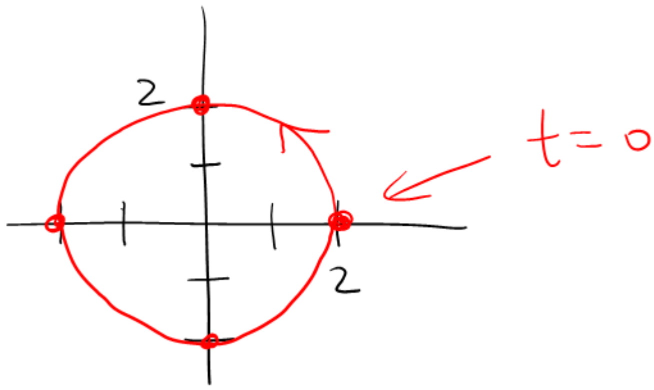
10.3 Parametric Curves and Calculus

Parametric curve example:

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t < 2\pi$$



FACT

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \text{if } \frac{dx}{dt} \neq 0}$$

Comes from Chain Rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\,\dfrac{dx}{dt}$

Ex: Find the slope of the tangent line at $t = 1$:

$$\begin{cases} x = 2t^2 + 1 \\ y = t^3 + t^5 \end{cases} \quad -\infty < t < \infty$$

$$\frac{dy}{dt} = 3t^2 + 5t^4$$

$$\frac{dx}{dt} = 4t$$

$$\frac{dy}{dx} = \frac{3t^2 + 5t^4}{4t}$$

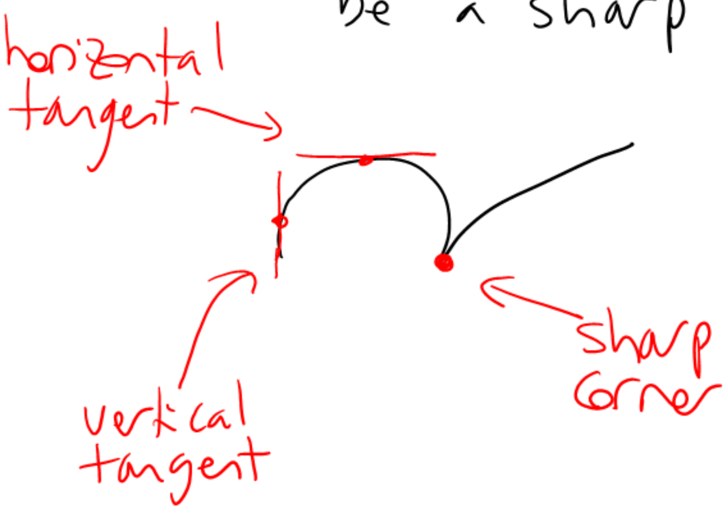
$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8}{4} = 2$$

FACT

If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ there is a horizontal tangent.

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ there is a vertical tangent.

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ there <u>may</u> be a sharp corner.



<u>Ex</u>:
$$\begin{cases} x = 4t + t^4 \\ y = 1 + t^2 \\ -\infty < t < \infty \end{cases}$$

Find all points (x,y) where there is a horizontal or vertical tangent.

$$\frac{dx}{dt} = 4 + 4t^3$$

$$4 + 4t^3 = 0$$

$$4t^3 = -4$$

$$t^3 = -1$$

$$t = -1$$

$$\frac{dy}{dt} = 2t$$

$$2t = 0$$

$$t = 0$$

Horizontal Tangent

$$\frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} \neq 0$$

$$\Rightarrow t = 0$$

$$\Rightarrow (x,y) = (0,1)$$

Vertical Tangent

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} \neq 0$$

$$\Rightarrow t = -1$$

$$\Rightarrow (x,y) = (-3,2)$$

The second derivative $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$

Note: Curve is <u>concave up</u> if $\dfrac{d^2y}{dx^2} > 0$ $\quad \cup$

" <u>concave down</u> if $\dfrac{d^2y}{dx^2} < 0$ $\quad \cap$

<u>Ex</u>: $\begin{cases} x = t^2 + 1 \\ y = t^6 + 5 \\ -\infty < t < \infty \end{cases}$

Find $\dfrac{d^2y}{dx^2}$

$$\frac{dy}{dt} = 6t^5$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^5}{2t} = 3t^4$$

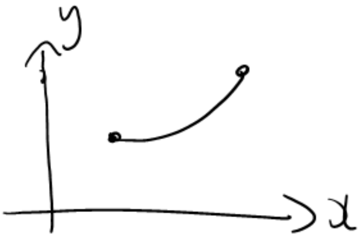
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\frac{d}{dt}\left(3t^4\right)}{2t}$$

$$= \frac{12t^3}{2t}$$

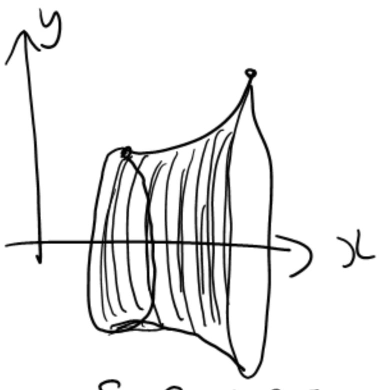
$$= 6t^2$$

Consider $\begin{cases} x = f(t) \\ y = g(t) \\ a \leq t \leq b \end{cases}$

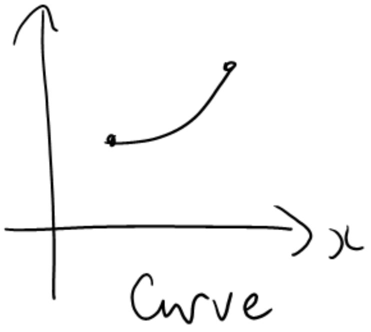


Arc Length $\quad s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

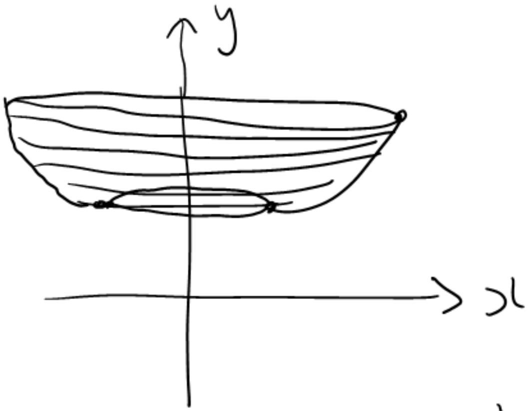
Revolve curve about x-axis



Surface Area $\quad S_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$



Revolve about y-axis :



SURFACE

Surface Area $S_y = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

<u>Ex</u>: Find the arc length of :

$$\begin{cases} x = t^2 \\ y = t^3 \\ 1 \le t \le 2 \end{cases}$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 \sqrt{t^2(4+9t^2)}\, dt$$

$$= \int_1^2 \sqrt{t^2}\sqrt{4+9t^2}\, dt$$

$$\boxed{\begin{array}{l} |t| \\ = t \text{ if } t>0 \end{array}}$$

$$= \int_1^2 t\sqrt{4+9t^2}\, dt$$

$$\boxed{\begin{array}{c} u = 4+9t^2 \\ du = 18t\, dt \\ \frac{du}{18} = t\, dt \\ t=1 \Rightarrow u=13 \\ t=2 \Rightarrow u=40 \end{array}}$$

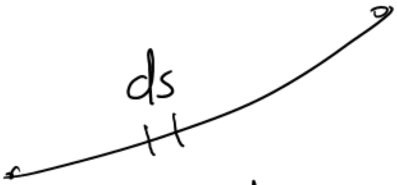
$$= \frac{1}{18}\int_{13}^{40} \sqrt{u}\, du$$

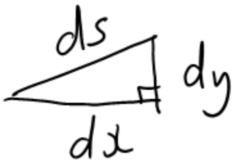
$$= \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{13}^{40}$$

$$= \frac{1}{18}\left(\frac{2}{3}\left(40^{3/2}\right) - \frac{2}{3}\left(13^{3/2}\right)\right)$$

$$= \frac{40^{3/2} - 13^{3/2}}{27}$$

ASIDE





$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: 10.3 # 13

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$

$$\begin{cases} x = 4\cos\theta \\ y = 4\sin\theta \end{cases} \qquad 0 \le \theta < 2\pi$$

$$\begin{cases} x = 4\cos t \\ y = 4\sin t \\ \quad 0 \le t < 2\pi \end{cases}$$

$$\frac{dy}{dt} = 4\cos t$$

$$\frac{dx}{dt} = -4\sin t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{4\cos t}{-4\sin t}$$

$$= -\cot t$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -1$$

$$\boxed{\tan\frac{\pi}{4} = 1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\frac{d}{dt}(-\cot t)}{-4\sin t}$$

$$= \frac{\csc^2 t}{-4\sin t}$$

$$= -\frac{1}{4}\csc^3 t$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\frac{1}{4}\left(\sqrt{2}\right)^3$$

$$= -\frac{2\sqrt{2}}{4}$$

$$= \frac{-\sqrt{2}}{2}$$

$$\boxed{\begin{aligned} \sin\frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\ \csc\frac{\pi}{4} &= \sqrt{2} \end{aligned}}$$