(Solvable)

**Comment:** To decide if a system is consistent, reduce it to REF.
To solve a system, reduce it to RREF.

**Definition:** Given $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, consider solutions to $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. If the only solution is $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly independent**. If there are solutions other than $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly dependent**.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$$

only solution is $c_1 = c_2 = \ldots = c_n = 0$

"Vectors are linearly <u>independent</u>"

$c_1 = c_2 = \ldots = c_n = 0$ and infinitely-many other solutions

"vectors are linearly <u>dependent</u>"

**Comment:** The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

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The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

**Comment:** a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$ is linearly dependent.

$$\underbrace{3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \end{bmatrix}}_{\text{at least one nonzero coefficient}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ are linearly dependent.

$$\underbrace{0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{\text{at least one nonzero coefficient}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are linearly dependent.

$$\underbrace{19\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{at least one nonzero coefficient}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Intuitive definition:</u>
Vectors are linearly dependent exactly when at least one of them can be written as a linear combination of the others.

**Example:** Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\end{bmatrix}$ linearly independent?

Let $c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

REF

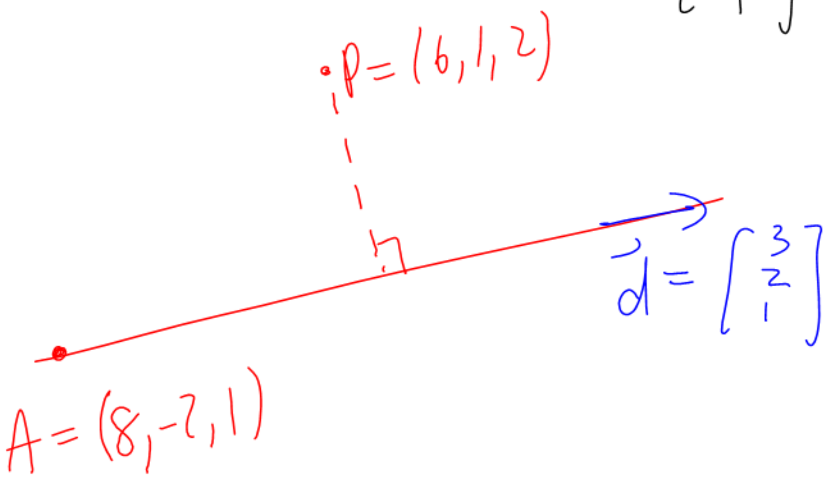
Unique solution

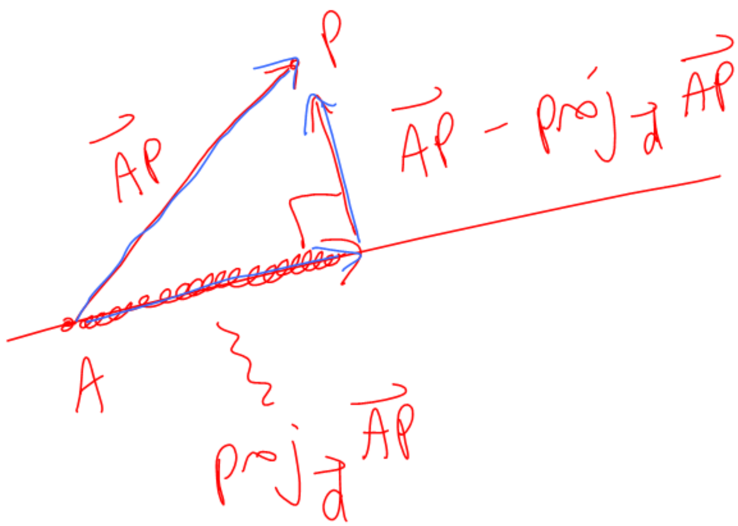
$$\begin{pmatrix} c_1 = 0 \\ c_2 = 0 \\ 2c_3 = 0 \Rightarrow c_3 = 0 \end{pmatrix}$$

$\boxed{\text{Yes}}$

# Test Review

**Ex:** Find the distance between $P = (6,1,2)$ and the line $\vec{x} = \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.




$$\vec{AP} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{proj}_{\vec{d}} \vec{AP} = \frac{\vec{d} \cdot \vec{AP}}{\|\vec{d}\|^2} \vec{d}$$

$$= \frac{1}{14}\begin{bmatrix}3\\2\\1\end{bmatrix}$$

$$\vec{AP} - \text{proj}_{\vec{d}}\,\vec{AP} = \frac{14}{14}\begin{bmatrix}-2\\3\\1\end{bmatrix} - \frac{1}{14}\begin{bmatrix}3\\2\\1\end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix}-31\\40\\13\end{bmatrix}$$

$$\text{distance} = \left\| \frac{1}{14}\begin{bmatrix}-31\\40\\13\end{bmatrix} \right\|$$

$$= \frac{1}{14}\left\| \begin{bmatrix}-31\\40\\13\end{bmatrix} \right\|$$

$$= \frac{\sqrt{2730}}{14}$$

Ex: Solve using Gauss-Jordan Elimination:

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 12 \\ 3 & 5 & 1 & 7 \\ 10 & 2 & 14 & 38 \end{array}\right]$$

$$\frac{R_1}{2} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 3 & 5 & 1 & 7 \\ 10 & 2 & 14 & 38 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 10R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 11 & -8 & -11 \\ 0 & 22 & -16 & -22 \end{array}\right]$$

$$\frac{R_2}{11} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -\frac{8}{11} & -1 \\ 0 & 22 & -16 & -22 \end{array}\right]$$

$$3 - \frac{16}{11} = \frac{33}{11} - \frac{16}{11}$$

$$\begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 22R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 17/11 & 4 \\ 0 & 1 & -8/11 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$$z = t$$

$$x + \frac{17}{11} z = 4 \implies x = 4 - \frac{17}{11} t$$

$$y - \frac{8}{11} z = -1 \implies y = -1 + \frac{8}{11} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -17/11 \\ 8/11 \\ 1 \end{bmatrix} t$$

Ex: Let $\vec{a} = [2,3]$ and $\vec{b} = [4,-6]$.
Find the angle between
$\vec{c} = \vec{a} + 7\vec{b}$ and $\vec{d} = \vec{a} - 2\vec{b}$.

$$\vec{c} = [2,3] + 7[4,-6]$$
$$= [2,3] + [28,-42]$$
$$= [30,-39]$$

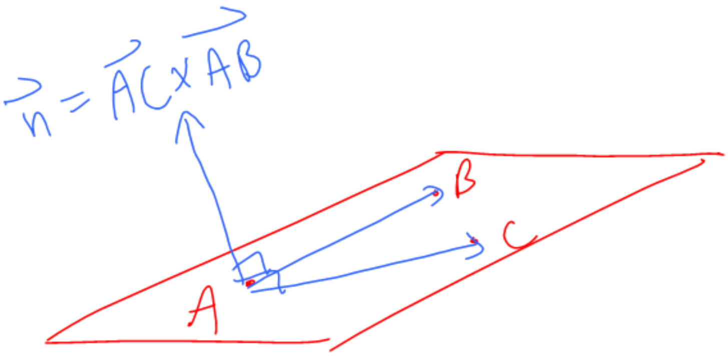
$$\vec{d} = [2,3] - 2[4,-6]$$
$$= [2,3] + [-8,12]$$
$$= [-6,15]$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$
$$\vec{c} \cdot \vec{d} = \|\vec{c}\| \, \|\vec{d}\| \cos\theta$$
$$-765 = \sqrt{2421}\sqrt{261} \cos\theta$$

$$\cos\theta = \frac{-765}{\left(\sqrt{2421}\sqrt{261}\right)}$$

$$\theta = \cos^{-1}\left(\frac{-765}{\left(\sqrt{2421}\sqrt{261}\right)}\right) \approx 164^\circ$$

Ex: Find the general form of the plane through $A = (8, 0, -2)$, $B = (-7, -40, -37)$ and $C = (3, -3, -3)$.




$$\vec{AB} = [-15, -40, -35]$$

$$\vec{AC} = [-5, -3, -1]$$

$$\vec{n} = \vec{AC} \times \vec{AB}$$

$$\boxed{\begin{array}{ccccc} -5 & -3 & -1 & -5 & -3 \\ & \times & \times & \times & \\ -15 & -40 & -35 & -15 & -40 \end{array}}$$

$$= [65, -160, 155]$$

Normal Form $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$$\begin{bmatrix} 65 \\ -160 \\ 155 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 65 \\ -160 \\ 155 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

General Form $\quad 65x - 160y + 155z = 210$

or $\quad -65x + 160y - 155z = -210$

Ex: Let $\vec{u} = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Find the value(s) of $x$ if:

a) $\vec{u}$ is parallel to $\vec{v}$

$$\vec{u} = k\vec{v}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = k\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

2nd component: $1 = k(3) \Rightarrow k = \frac{1}{3}$

1st component: $x = k(4) \Rightarrow x = \frac{4}{3}$

b) $\vec{u}$ is perpendicular to $\vec{v}$

$$\vec{u} \cdot \vec{v} = 0$$

$$4x + 3 = 0$$

$$4x = -3$$

$$x = -\frac{3}{4}$$

c) $\vec{u}$ has length $\sqrt{8}$

$$\|\vec{u}\| = \sqrt{8}$$

$$\sqrt{x^2 + 1} = \sqrt{8}$$

$$x^2 + 1 = 8$$

$$x^2 = 7$$

$$x = \pm\sqrt{7}$$

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ASIDE

Area of parallelogram determined by:

a) [2, 3] and [4, 19]

$$\left|\begin{vmatrix} 2 & 3 \\ 4 & 19 \end{vmatrix}\right|$$

b) [2, 3, 1] and [4, 19, 2]

$$\| [2,3,1] \times [4,19,2] \|$$