**Example:** Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + 2y + 3z &= 7 \\ 3x + 3y + 3z &= 15 \\ 5x + 7y + 9z &= 29 \end{aligned}$$

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 3 & 3 & 3 & 15 \\ 5 & 7 & 9 & 29 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -3 & -6 & -6 \\ 0 & -3 & -6 & -6 \end{array}\right]$$

$$\frac{R_2}{-3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & -6 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

Circle the leading nonzero entry in each row.
Any column without a circle gets a parameter.

free variable $\rightarrow$ $z = t$ $\leftarrow$ parameter

$x - z = 3 \Rightarrow x = 3 + z \Rightarrow x = 3 + t$ ✓

$y + 2z = 2 \Rightarrow y = 2 - 2z \Rightarrow y = 2 - 2t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$ ✓

**Example:** Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + \ y - \ \ 6z &= 17 \\ 2x + 2y - \ \ 8z &= 22 \\ 3x + 3y - 14z &= 39 \end{aligned}$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 2 & 2 & -8 & 22 \\ 3 & 3 & -14 & 39 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 4 & -12 \end{array}\right]$$

Can't make this 1. Move further right.

$$\frac{R_2}{4} \left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 4 & -12 \end{array}\right]$$

$$\begin{array}{l} R_1 + 6R_2 \\ \\ R_3 - 4R_2 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{RREF}$$

$$y = k$$

$$x + y = -1 \implies x = -1 - y \implies x = -1 - k$$

$$z = -3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k$$

**Example:** Solve by Gauss-Jordan Elimination:

$$\begin{aligned} w + \ x + 2y + 10z &= 5 \\ x + \ y + \ \ z &= 2 \\ w + 3x + 4y + 12z &= 9 \end{aligned}$$

$$\begin{array}{cccc|c} w & x & y & z & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 10 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 3 & 4 & 12 & 9 \end{array}\right]$$

$$R_3 - R_1 \quad \left[\begin{array}{cccc|c} 1 & 1 & 2 & 10 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ R_3 - 2R_2 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 9 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$y = s \qquad z = t$

$w + y + 9z = 3 \Rightarrow w = 3 - y - 9z \Rightarrow w = 3 - s - 9t$

$x + y + z = 2 \Rightarrow x = 2 - y - z \Rightarrow x = 2 - s - t$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -9 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$$

**Example:** Find the intersection of the two lines:

$$\vec{x} = \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Set $\vec{x} = \vec{x}$

$$\begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix}$$

$$s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -6 \end{bmatrix}$$

$$\begin{cases} 2s - t = 0 \\ s - t = -2 \\ -s - t = -6 \end{cases}$$

$$\begin{array}{cc} s & t \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -6 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$
$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & -1 & -6 \end{array}\right]$$

$$\begin{array}{cc} s & t \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right] \quad \text{(check this)}$$

$s = 2, \; t = 4$

Sub $s=2$ into 1st equation OR sub $t=4$ into 2nd equation:

$$\vec{x} = \begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix}$$

**Example:** How many solutions does the following system have?

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$$

$$\begin{array}{cc} x & y \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right]$$

$$R_2 - kR_1 \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

$1-k^2 \neq 0$

$$\frac{R_2}{1-k^2} \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{array}\right]$$

REF

$\boxed{\text{1 solution}}$

$1-k^2 = 0$

$(1+k)(1-k) = 0$

$k=-1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

$\boxed{\text{no solution}}$

$k=1$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$y=k$

$\boxed{\infty\text{-many Solutions}}$

$$\begin{cases} \text{1 solution when } 1-k^2 \neq 0 \\ \text{no solution when } k=-1 \\ \infty\text{-many solutions when } k=1 \end{cases}$$