Definition: A **matrix** is a rectangular array of numbers. For example, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Definition: The **determinant** of a matrix A is written $\det A$ or $|A|$. The determinant is only defined for square matrices.

Fact:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

AND

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Comment: The second formula is called **cofactor expansion**.

Comment: Notice that the second term in the cofactor expansion has a negative sign.

Example: Compute $\det \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$

$$= 1 \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 0 & 7 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix}$$

$$= 1(-11) - 4(14) + 6(12)$$

$$= 5$$

Example: Compute $\begin{vmatrix} -1 & -4 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 8 \end{vmatrix}$

$$= -1\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -1(6) + 4(6) + 6(0)$$

$$= 18$$

Notation: Let:
$\vec{i} = [1, 0, 0]$
$\vec{j} = [0, 1, 0]$
$\vec{k} = [0, 0, 1]$

Fact: A second method of calculating the cross product is:

$$[u_1, u_2, u_3] \times [v_1, v_2, v_3] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example: Calculate $[2, 1, 3] \times [-6, 4, 2]$ using the original method.

$[-10, -22, 14]$

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 \\ -6 & 4 & 2 & -6 & 4 \end{matrix}$$

Example: Calculate $[2, 1, 3] \times [-6, 4, 2]$ using the second method. Notice why cofactor expansion has a negative sign on the second term.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ -6 & 4 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 3 \\ -6 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix}$$

$$= \vec{i}(-10) - \vec{j}(22) + \vec{k}(14)$$

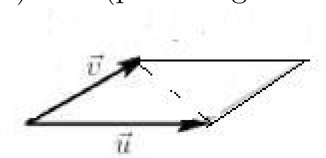
$$= -10[1,0,0] - 22[0,1,0] + 14[0,0,1]$$

$$= [-10, -22, 14]$$

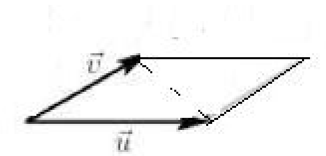
The $\vec{j}$ term is negative to correct for columns being backwards in $\begin{vmatrix} 2 & 3 \\ -6 & 2 \end{vmatrix}$.

Fact: Three geometry formulas:

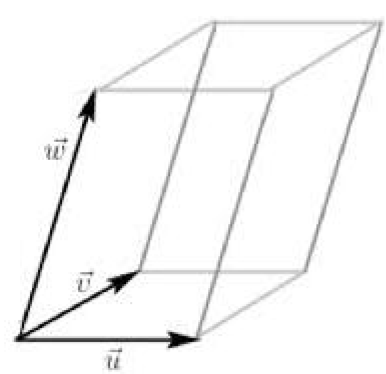
1) Area(parallelogram in $\mathbb{R}^3$)=$||\vec{u} \times \vec{v}||$



2) Area(parallelogram in $\mathbb{R}^2$)= absolute value of det $\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$



3) Volume(parallelepiped in $\mathbb{R}^3$)= absolute value of det $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$



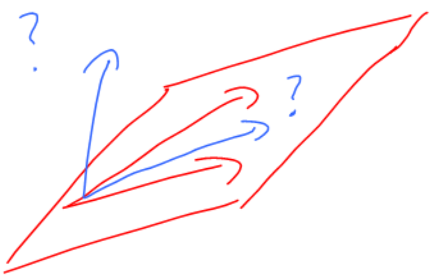
parallelepiped (slanted box)

Example: Find the area of the parallelogram determined by [1, 6] and [3, 5].

$$\left| \begin{vmatrix} 1 & 6 \\ 3 & 5 \end{vmatrix} \right| = |-13| = 13$$

absolute value

Example: Do the vectors [1, 4, 7], [2, 5, 9] and [1, −2, −3] lie in a common plane?



(exactly when)

Yes if and only if V(parallelepiped) $= 0$.

$$V(\text{parallelepiped}) = \left| \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 9 \\ 1 & -2 & -3 \end{vmatrix} \right|$$

absolute value

$$= \left| 1 \begin{vmatrix} 5 & 9 \\ -2 & -3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 9 \\ 1 & -3 \end{vmatrix} + 7 \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} \right|$$

$$= \left| 1(3) - 4(-15) + 7(-9) \right|$$

$$= |0|$$

$$= 0$$

$$\boxed{\text{Yes}}$$

Chapter 2: Systems of Linear Equations

2.1 Linear Systems

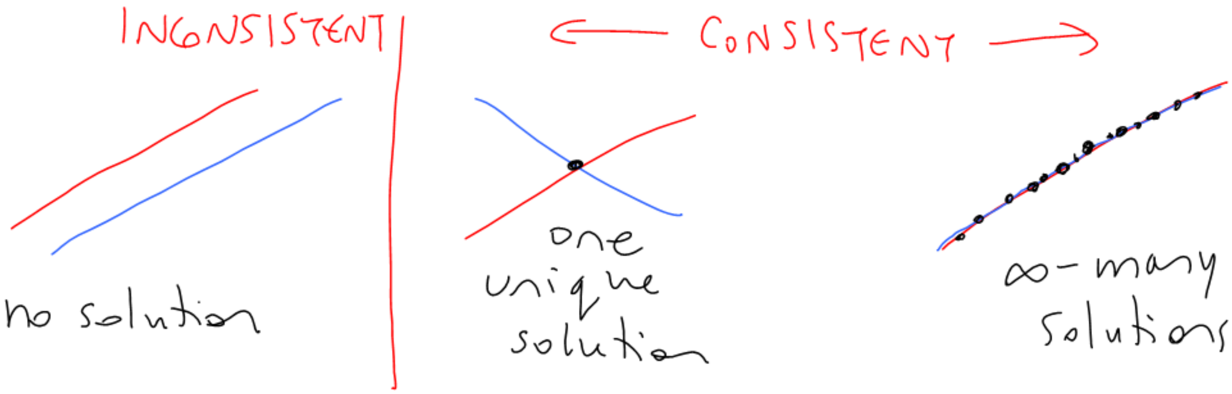
Definition: A **linear equation** in $\mathbb{R}^2$ has the form $ax + by = c$, where a, b and c are real numbers.

Definition: A **linear system** in $\mathbb{R}^2$ consists of two or more linear equations. It's often just called a **system**.

Comment: Here's an example of a system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

Fact: A system can have: no solution, one unique solution or infinitely-many solutions.



Definition: A system with no solution is called an **inconsistent system**.
A **consistent system** has one solution or infinitely-many solutions. In other words, a consistent system is solvable.

Definition: Consider the system:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

The matrix $\begin{bmatrix} 2 & 6 \\ -3 & 3 \end{bmatrix}$ is called the **coefficient matrix**.

The matrix $\left[\begin{array}{cc|c} 2 & 6 & -14 \\ -3 & 3 & -15 \end{array}\right]$ is called the **augmented matrix**.

Fact: There are three types of elementary row operations that can be performed on an augmented matrix. These row operations don't change the solution of the system:
1) Swap two rows
2) Multiply or divide a row by a nonzero real number
3) (Current Row) ± #(Pivot Row)

Example: Solve by elimination:

$$\begin{aligned} 2x + 6y &= -14 \\ -3x + 3y &= -15 \end{aligned}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & \# \end{array} \\ \left[\begin{array}{cc|c} 2 & 6 & -14 \\ -3 & 3 & -15 \end{array}\right] \end{array}$$

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ -3 & 3 & -15 \end{array}\right]$$

$$R_2 + 3R_1 \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 12 & -36 \end{array}\right]$$

$$\vdots$$

$$x =$$

$$y =$$

To Be Continued