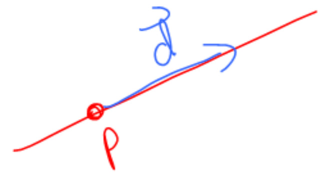
**Part 2. Lines in $\mathbb{R}^3$**

**Example:** Consider the line through $P = (2, 1, 12)$ and $Q = (0, -3, 6)$. Describe the line in both vector and parametric form.

direction vector $\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$

vector form $\vec{x} = \vec{p} + t\vec{d}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

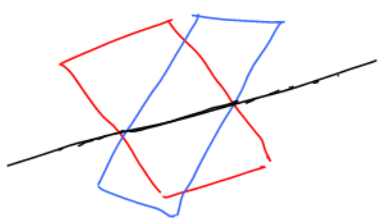



parametric form $\begin{cases} x = 2 - 2t \\ y = 1 - 4t \\ z = 12 - 6t \end{cases}$

**Definition:** A **plane** is an infinite flat surface.

**Fact:** $ax + by + cz = d$ is the general form for a plane in $\mathbb{R}^3$.

**Comment:** General form for a line in $\mathbb{R}^3$ is inconvenient so we will omit it. It would consist of two equations, describing the intersection of two planes.

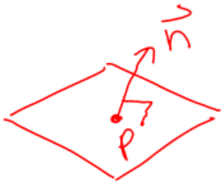


$$\begin{cases} 2x - 3y + 4z = 6 \\ 3x + 4y + 7z = 11 \end{cases}$$

**Comment:** Similarly we omit normal form for a line in $\mathbb{R}^3$.

**Part 3. Planes in $\mathbb{R}^3$**

**Example:** Consider the plane through $P = (1, -1, 3)$ with normal $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Describe the plane in both normal and general form.



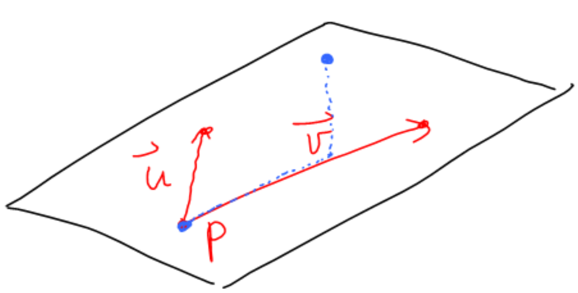

normal form $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

general form $\quad x + y + 2z = 6$

**Definition:** The **vector form** for a plane in $\mathbb{R}^3$ is $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ where:
$\vec{u}$ and $\vec{v}$ are nonparallel direction vectors
$s$ and $t$ represent any real numbers

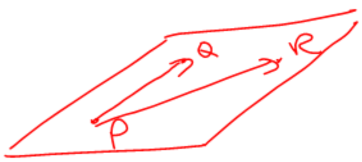
vectorization of any point on the plane $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$



**Example:** Consider the plane through $P = (6, 0, 0)$, $Q = (0, 6, 0)$ and $R = (0, 0, 3)$. Describe the plane in vector and parametric form.

two nonparallel direction vectors $\vec{u} = \overrightarrow{PQ} = \begin{bmatrix} -6 \\ 6 \\ 0 \end{bmatrix}$

$\vec{v} = \overrightarrow{PR} = \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$




Vector form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 6 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$$

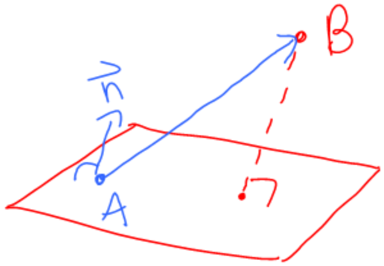
parametric form
$$\begin{cases} x = 6 - 6s - 6t \\ y = 6s \\ z = 3t \end{cases}$$

**Example:** Summarize the twelve descriptions

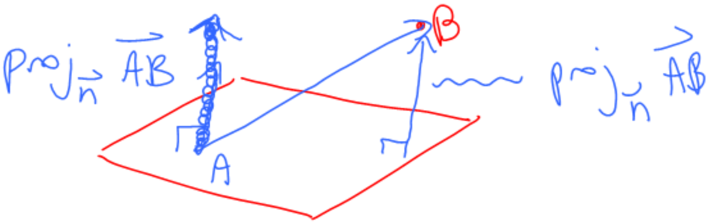
| | Lines in $\mathbb{R}^2$ | Lines in $\mathbb{R}^3$ | Planes in $\mathbb{R}^3$ |
|---|---|---|---|
| General | $ax + by = c$ | OMIT | $ax + by + cz = d$ |
| Normal | $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ | OMIT | $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ |
| Vector | $\vec{x} = \vec{p} + t\vec{d}$ | $\vec{x} = \vec{p} + t\vec{d}$ | $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ |
| Parametric | $\begin{cases} x = \\ y = \end{cases}$ | $\begin{cases} x = \\ y = \\ z = \end{cases}$ | $\begin{cases} x = \\ y = \\ z = \end{cases}$ |

## Part 4. Geometry Problems

**Example:** Find the distance between $B = (1, 3, 3)$ and the plane $\mathcal{P} : x + y + 2z = 7$




Let $A$ = any point on plane




$$\text{distance} = \left\| \text{proj}_{\vec{n}} \overrightarrow{AB} \right\|$$

$$A = (7, 0, 0)$$

$$\overrightarrow{AB} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

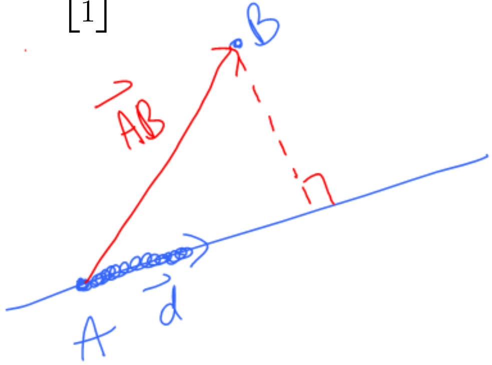
$$\text{proj}_{\vec{n}} \overrightarrow{AB} = \frac{\vec{n} \cdot \overrightarrow{AB}}{\|\vec{n}\|^2} \vec{n}$$

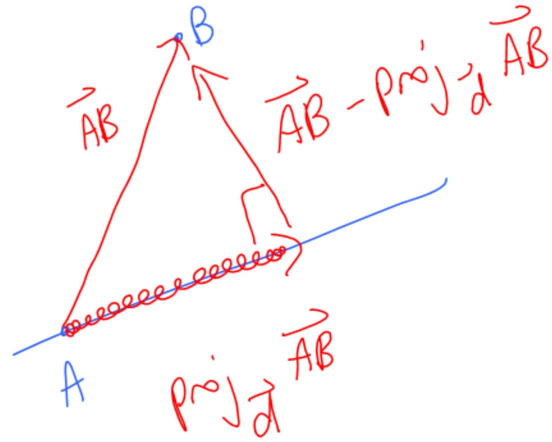
$$= \frac{3}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\| = \frac{\sqrt{6}}{2}$$

**Example:** Find the distance between $B = (1, 1, 0)$ and the line through $A = (0, 1, 2)$ with $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.




$$\text{distance} = \| \vec{AB} - \text{proj}_{\vec{d}} \vec{AB} \|$$

$$\vec{AB} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

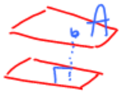
$$\text{proj}_{\vec{d}} \vec{AB} = \frac{\vec{d} \cdot \vec{AB}}{\| \vec{d} \|^2} \vec{d}$$

$$= \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

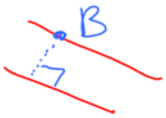
$$\vec{AB} - \text{proj}_{\vec{d}} \vec{AB} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix} \quad \text{or} \quad \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \frac{3}{2} \left\| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \frac{3\sqrt{2}}{2}$$

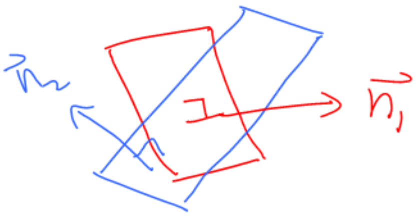


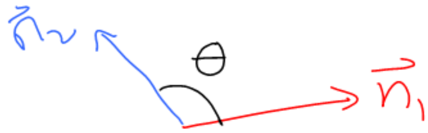

**Comment:** To find the distance between parallel planes, pick a point on one of the planes. Find the distance between that point and the other plane.




**Comment:** To find the distance between parallel lines, pick a point on one of the lines. Find the distance between that point and the other line.

**Definition:** The **angle between planes** is defined as the angle between their normals.




**Definition:** **Parallel planes** have parallel normals.
**Perpendicular planes** have perpendicular normals.

Write an equation that describes
" $\vec{n}_1$ is parallel to $\vec{n}_2$ "

$$\vec{n}_1 = k\,\vec{n}_2$$

Write an equation that describes
" $\vec{n}_1$ is perpendicular to $\vec{n}_2$ "

$$\vec{n}_1 \cdot \vec{n}_2 = 0$$