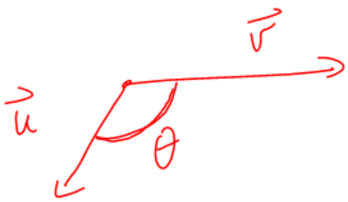
List of HW Problems and Full Solutions
are on website.

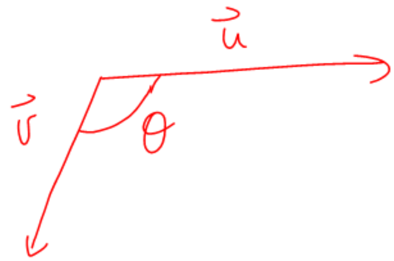
Problems are on D2L.

Do HW for Sections 1.1 and 1.2

No formula sheet for Math 251.

Fact: Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^n$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0^\circ \leq \theta \leq 180^\circ$





Fact: For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $\quad \vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$

Comment: In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ.

Comment: In the special case where $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u} \cdot \vec{v}$ gives the value of $\cos\theta$.

Example: Find the angle between $\vec{u} = [1, -4]$ and $\vec{v} = [2, 3]$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$-10 = \sqrt{17}\sqrt{13} \cos\theta$$

$$\frac{-10}{\sqrt{17}\sqrt{13}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-10}{(\sqrt{17}\sqrt{13})}\right)$$

$$\theta \approx 132^\circ$$

Example: If $0^\circ \leq \theta < 90^\circ$, what is the sign of $\vec{u} \cdot \vec{v}$?
What if $\theta = 90^\circ$?
What if $90^\circ < \theta \leq 180^\circ$?

$0^\circ \leq \theta < 90^\circ$	$\theta = 90^\circ$	$90^\circ < \theta \leq 180^\circ$
$\Rightarrow \cos\theta > 0$	$\Rightarrow \cos\theta = 0$	$\Rightarrow \cos\theta < 0$
$\Rightarrow \vec{u} \cdot \vec{v} > 0$	$\Rightarrow \vec{u} \cdot \vec{v} = 0$	$\Rightarrow \vec{u} \cdot \vec{v} < 0$

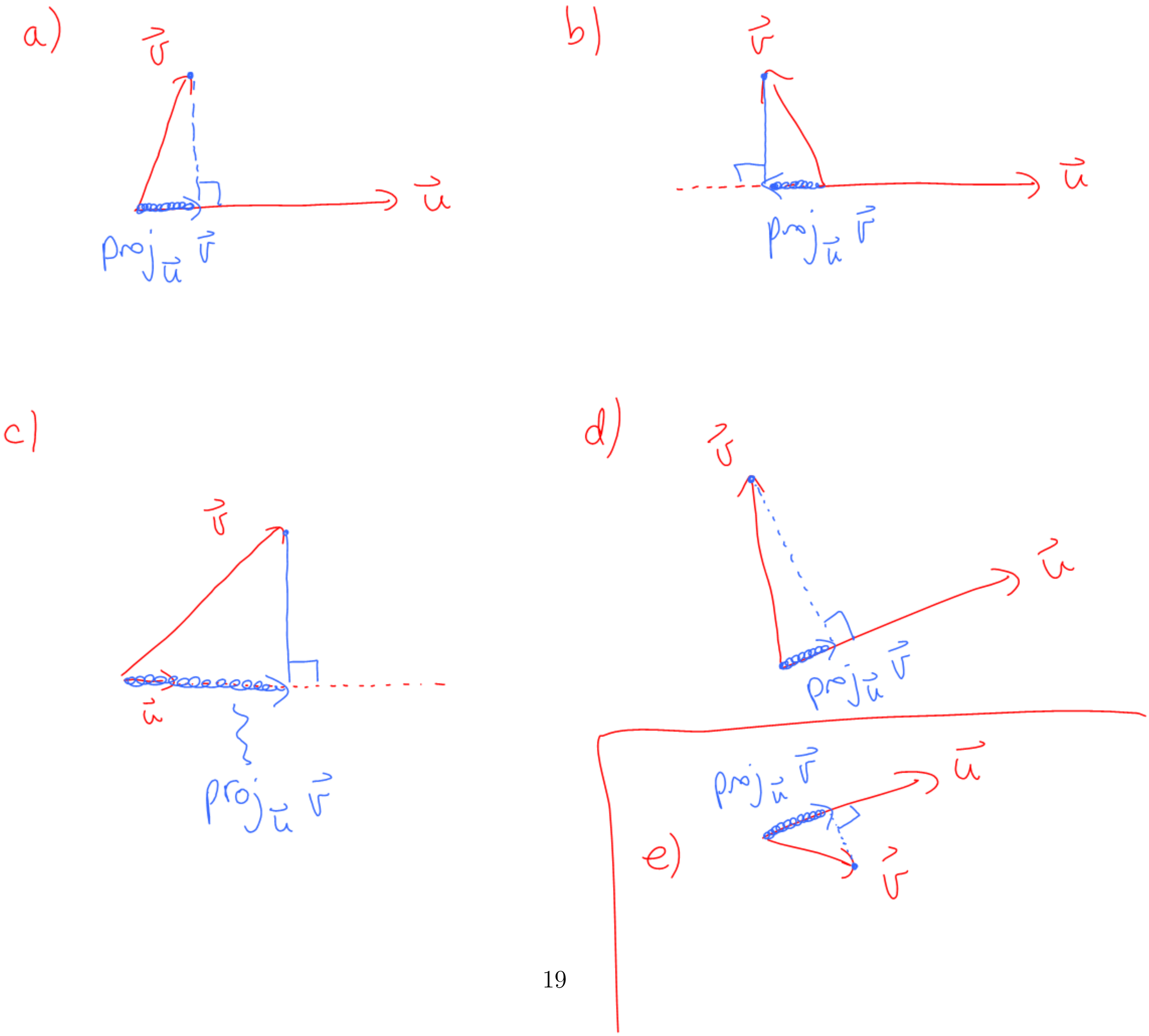
Definition: Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$.

Comment: The following statements are equivalent in 2D and 3D:
Vectors $\vec{u}$ and $\vec{v}$ are perpendicular (geometry language)
Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (algebra language)

Comment: In higher dimensions it's more appropriate to use the word **orthogonal** rather than perpendicular.

Definition: The **projection** of $\vec{v}$ onto $\vec{u}$ is written $\text{proj}_{\vec{u}}\vec{v}$. This could be read as the projection onto $\vec{u}$ of $\vec{v}$.

Example: Let's draw a few instances of $\text{proj}_{\vec{u}}\vec{v}$

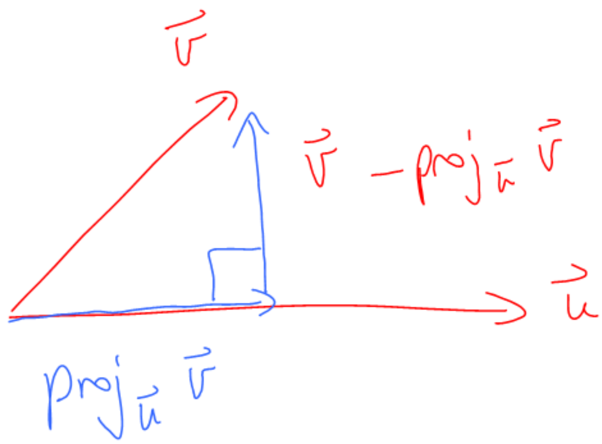


Fact: $\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$

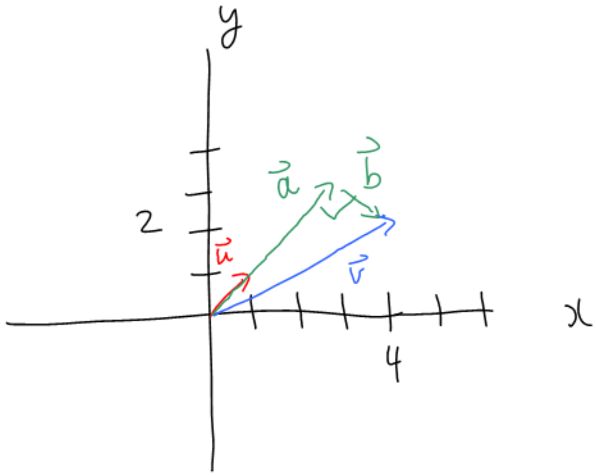
Example: Find $\text{proj}_{\vec{u}}\vec{v}$ for $\vec{u} = [1, 2]$ and $\vec{v} = [1, 3]$

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$$
$$= \frac{7}{5}[1, 2] \quad \text{or} \quad \frac{7}{5}\vec{u}$$

Fact: Given vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, there is exactly one way to decompose $\vec{v}$ into two vectors that are parallel and perpendicular to $\vec{u}$.



Example: Let $\vec{u} = [1,1]$ and $\vec{v} = [4,2]$. Find vectors $\vec{a}$ and $\vec{b}$ so that $\vec{v} = \vec{a} + \vec{b}$, $\vec{a}$ is parallel to $\vec{u}$, and $\vec{b}$ is perpendicular to $\vec{u}$.



$$\vec{a} = \mathrm{proj}_{\vec{u}} \vec{v}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

$$= \frac{6}{2} [1,1]$$

$$= 3 [1,1]$$

$$= [3,3]$$

$$\vec{a} + \vec{b} = \vec{v}$$

$$\vec{b} = \vec{v} - \vec{a}$$

$$\vec{b} = [4,2] - [3,3]$$

$$\vec{b} = [1,-1]$$

1.3 Lines and Planes

Part 1. Lines in $\mathbb{R}^2$

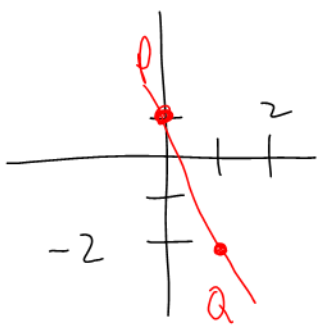
Definition: The **general form** of a line in $\mathbb{R}^2$ is $ax + by = c$

Example: Consider the line $3x + y = 1$. Find two points on the line and graph the line.

Set $x=0$: $y=1$ $\qquad P=(0,1)$

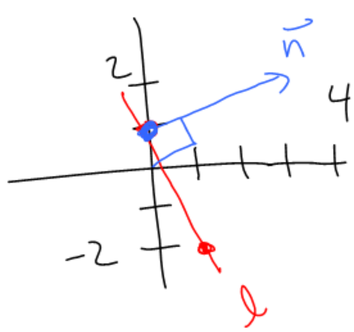
Set $x=1$: $3+y=1$

$y=-2$ $\qquad Q=(1,-2)$



Definition: A **normal vector** is orthogonal to a given line. It is written $\vec{n}$. Its components are the coefficients from the general form.

For the line above, $\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



Definition: The **normal form** of a line in $\mathbb{R}^2$ is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{p}$ is the vectorization of any point on the line.

Example: Describe the line $3x + y = 1$ in normal form. Show that expanding normal form gives general form.

$$\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

normal form

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

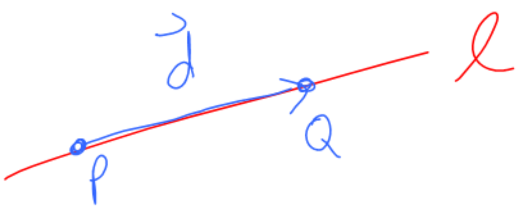
any point on the line

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

general form

$$3x + y = 1$$

Definition: A **direction vector** for a line is $\vec{d} = \overrightarrow{PQ}$, where P and Q are any two points on the line.



Definition: The **vector form** for a line in $\mathbb{R}^2$ is $\vec{x} = \vec{p} + t\vec{d}$, where t represents any real number.

$\vec{x}$: $\begin{bmatrix} x \\ y \end{bmatrix}$

t: real #

$\vec{d}$: $\overrightarrow{PQ}$

$\vec{p}$: vectorization of any point on the line

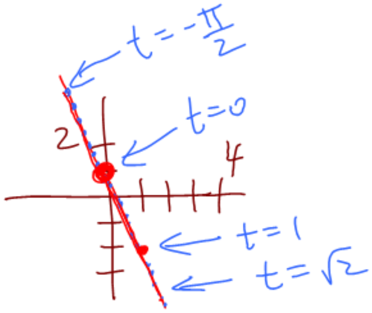
Example: Describe the line $3x + y = 1$ in vector form. Show that as t varies, the line is traced out.

2 points on the line : $P = (0, 1)$
$Q = (1, -2)$

direction vector $\vec{d} = \overrightarrow{PQ}$
$= \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ Think $Q - P$

vector form $\vec{x} = \vec{p} + t\vec{d}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$



Definition: The **parametric form** for a line in $\mathbb{R}^2$ is:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

Example: Describe the line $3x + y = 1$ in parametric form.

Vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Expand: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} t \\ -3t \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 1-3t \end{bmatrix}$

parametric form $\begin{cases} x = t \\ y = 1-3t \end{cases}$

Comment: A given line can be described in a specific form in multiple ways, for example $3x + y = 1$ and $6x + 2y = 2$ are general forms for the same line.

Example: Summarize the four forms of a line in $\mathbb{R}^2$

General Form $3x + y = 1$

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

Normal Form $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Vector Form $\vec{x} = \vec{p} + t\vec{d}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Parametric Form $\begin{cases} x = t \\ y = 1 - 3t \end{cases}$