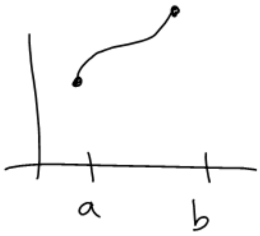
26.6 Cont'd

Arc Length from $x=a$ to $x=b$



$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Ex: Path of a plane $y=\frac{2}{3}(x^2-1)^{3/2}$ for $x \geq 1$.
Find the length of the path for $1 \leq x \leq 3$.

$$\frac{dy}{dx} = \frac{2}{3}\cdot\frac{3}{2}(x^2-1)^{1/2}(2x)$$

$$= \sqrt{x^2-1}\,(2x)$$

$$\left(\frac{dy}{dx}\right)^2 = (x^2-1)(4x^2)$$

$$= 4x^4-4x^2$$

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{4x^4-4x^2+1}\, dx$$

$$= \int_1^3 \sqrt{(2x^2-1)^2}\, dx$$

$$= \int_1^3 |2x^2-1|\, dx$$

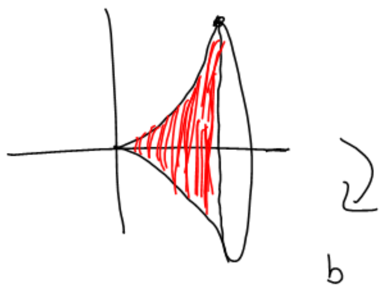
$$= \int_1^3 (2x^2-1)\, dx$$

$$= \left[\frac{2x^3}{3} - x\right]_1^3$$

$$= (18 - 3) - \left(\frac{2}{3} - 1\right)$$

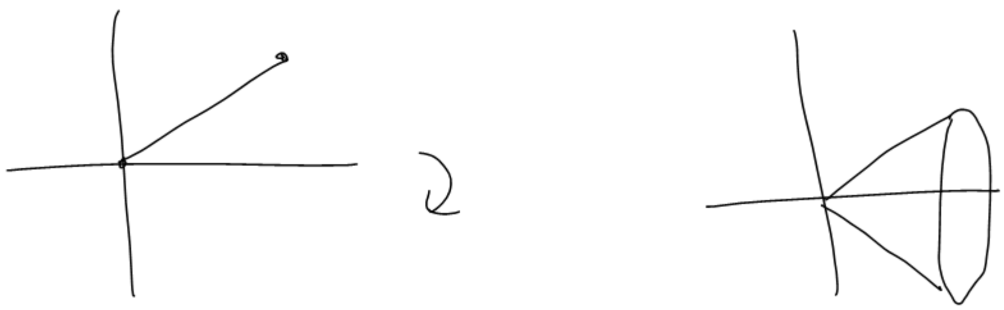
$$= \frac{46}{3}$$

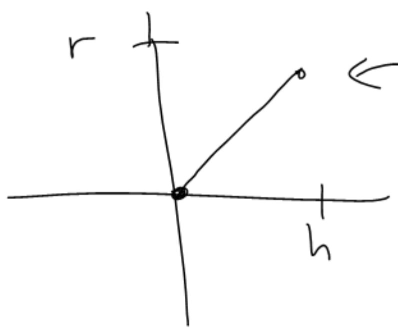
(Lateral) Surface Area of a Solid of Revolution about x-axis



$$SA = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

<u>Ex</u>: Find (lateral) surface area of a cone with height h and radius r.





$$y = mx + b$$

$$y = \frac{r}{h}x$$

$$\frac{dy}{dx} = \frac{r}{h}$$

$$SA = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^h \frac{r}{h} x \sqrt{1+\left(\frac{r}{h}\right)^2}\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1+\left(\frac{r}{h}\right)^2} \int_0^h x\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1+\left(\frac{r}{h}\right)^2} \left[\frac{x^2}{2}\right]_0^h$$

$$= \cancel{2}\pi \frac{r}{\cancel{h}} \sqrt{1+\frac{r^2}{h^2}} \quad \frac{\cancel{h^2}\, h}{\cancel{2}}$$

$$= \pi r h \sqrt{1+\frac{r^2}{h^2}}$$

$$\sqrt{h^2}$$

$$= \pi r \sqrt{h^2 + r^2}$$

Practice Problem #29

Find $\int_0^1 x^3 (2x^4+1)^5\, dx$

$u = 2x^4 + 1$

$du = 8x^3\, dx$

$\frac{1}{8} du = x^3\, dx$

if $x=0$, $u=1$

if $x=1$, $u=3$

$$= \frac{1}{8}\int_1^3 u^5\, du$$

$$= \frac{1}{8}\left(\frac{u^6}{6}\right)\Big|_1^3$$

$$= \frac{1}{48} u^6 \Big|_1^3$$

$$= \frac{1}{48}\left(3^6 - 1\right)$$

$$= \frac{91}{6}$$

Alternatively:

$$\int_0^1 x^3(2x^4+1)^5\, dx = \frac{1}{8}\int_{x=0}^{x=1} u^5\, du$$

$$= \frac{1}{8}\left(\frac{u^6}{6}\right)\Big|_{x=0}^{x=1}$$

$$= \frac{1}{48} u^6 \Big|_{x=0}^{x=1}$$

$$= \frac{1}{48}\left(2x^4+1\right)^6\Big|_0^1$$

$$= \frac{1}{48}\left(3^6 - 1\right)$$

$$= \frac{91}{6}$$