14.7 Stokes' Theorem

Stokes' Theorem

Def

A surface is <u>oriented</u> if it has 2 sides

e.g. A Möbius strip is <u>not</u> oriented

<u>Stokes' Theorem</u>

Let S be an oriented surface in 3D with positively-oriented boundary curve C. Then

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

Physical Interpretation:

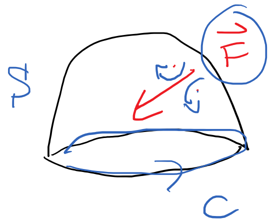
Let $\vec{F}$ = force field

$$\iint_{S} (\text{curl}\,\vec{F}) \cdot \vec{n}\, d\vec{S} = \sum_{\substack{\infty\text{-many} \\ \text{tiny areas } d\vec{S}}} \left(\begin{matrix}\text{magnitude of curl } \vec{F} \\ \text{in direction } \perp \text{ to } S\end{matrix}\right) \cdot d\vec{S}$$

$$= c_1\, d\vec{S} - c_2\, d\vec{S} + c_3\, d\vec{S} - c_4\, d\vec{S}$$

c_i : non-negative

lots of cancellation



= net effect on C

= Work done by $\vec{F}$ on C

$$= \oint_{C} \vec{F} \cdot \vec{T}\, ds$$

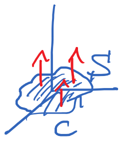
$\vec{F}$:	N
curl $\vec{F}$:	$\frac{N}{m}$
$d\vec{S}$:	m^2
right side of Stokes' Equation:	$N \cdot m$

Comparison:

Stokes' Theorem (3D)

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\mathrm{curl}\,\vec{F})\cdot\vec{n}\,dS$$

Suppose S and C are in the xy-plane



$\Rightarrow$ upper unit normal $\vec{n} = [0, 0, 1]$

$\Rightarrow$ $\mathrm{curl}\,\vec{F}\cdot\vec{n} = \left[\quad , \quad , \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]\cdot[0, 0, 1]$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Green's Theorem (2D)

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

" Green's Theorem is 2D Stokes' Theorem"

WORK	FLUX
Green's Thm (2D)	2D Divergence Thm
Stokes' Thm (3D)	3D "

$\underline{Ex}$: C: positively-oriented intersection of $x^2+y^2=1$ and $y+z=4$

$\vec{F} = [-y,\ xy,\ z^2]$

Use Stokes' Theorem to evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

S: $y+z=4$
$z=4-y$

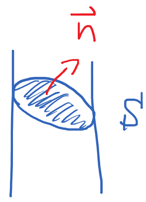
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & xy & z^2 \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\,(y+1)$$

$$= [0,\ 0,\ y+1]$$

$$\vec{n}\,dS = \pm[-z_x,\ -z_y,\ 1]\,dy\,dx$$

$$= \pm[0,\ 1,\ 1]\,dy\,dx$$



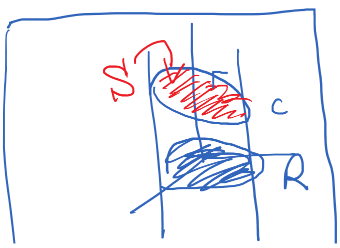
Upwards $\Rightarrow$ $z>0$ $\Rightarrow$ Choose $+$

$$\vec{n}\,dS = [0,1,1]\,dy\,dx$$

$$\boxed{\begin{aligned}(\text{curl}\,\vec{F})\cdot\vec{n}\,dS &= [0,0,y+1]\cdot[0,1,1]\,dy\,dx \\ &= (y+1)\,dy\,dx\end{aligned}}$$

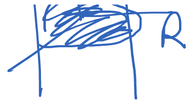
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dS$$

$$= \iint_R (y+1)\,dy\,dx$$



$\iint\limits_R (y+1 \ldots$

$$= \int_0^{2\pi}\int_0^1 \underbrace{(r\sin\theta + 1)\, r}_{r^2\sin\theta + r}\, dr\, d\theta$$



$x^2+y^2=1$

$r^2=1$

$\cancel{r=\pm 1}$ $\quad r=1$

$R: \quad 0 \le r \le 1$
$\quad\; 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta + \frac{r^2}{2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{\sin\theta}{3} + \frac{1}{2}\right) d\theta$$

$$= \left[\frac{-\cos\theta}{3} + \frac{\theta}{2}\right]_0^{2\pi}$$

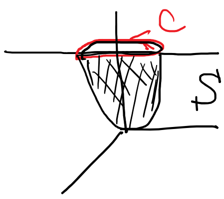
$$= \left(-\frac{1}{3} + \pi\right) - \left(-\frac{1}{3}\right)$$

$$= \pi$$

Ex: Use Stokes' Theorem to evaluate

$\iint\limits_S (\text{curl}\,\vec{F})\cdot\vec{n}\, dS$ where $\vec{F} = [-2y,\ 7z,\ x]$

and S: part of $z = x^2+y^2$ that lies below $z=4$.



$z=4$

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\, ds = \iint\limits_S (\text{curl}\,\vec{F})\cdot\vec{n}\, dS}$$

$C: \quad z = z$

$x^2 + y^2 = 4$

$r^2 = 4$

$\cancel{r = \pm 2} \quad r = 2$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = 4$

$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = 4$

$$C: \boxed{\begin{array}{lll} x = 2\cos t & y = 2\sin t & z = 4 \\ dx = -2\sin t\,dt & dy = 2\cos t\,dt & dz = 0 \\ & 0 \le t \le 2\pi & \end{array}}$$

$$\iint_S (\operatorname{Curl}\vec{F}) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot \vec{T}\, ds$$

$$= \oint_C (P\,dx + Q\,dy + R\,dz)$$

$$= \oint_C (-2y\,dx + 7z\,dy + x\,dz)$$

$$= \int_0^{2\pi} [-4\sin t\,(-2\sin t\,dt) + 28(2\cos t\,dt)]$$

$$= \int_0^{2\pi} [\underbrace{\cancel{8\sin^2 t}}_{4 - 4\cos 2t} + 56\cos t]\,dt$$

$$\boxed{8\sin^2 t = (1 - \cos 2t)4}$$

$$= \left[4t - 2\sin 2t + 56\sin t\right]_0^{2\pi}$$

$$= 8\pi$$