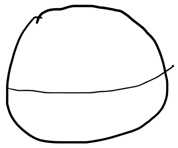
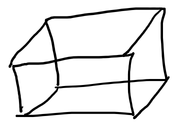
14.6 The 3D Divergence Theorem

A closed surface : bounds a solid region

Closed surfaces :





Not closed :

3D Divergence Theorem :

Let S be a closed surface that bounds a solid region Q.

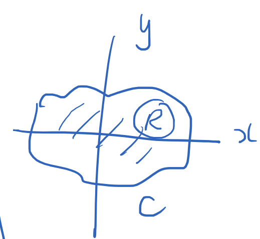
$$\oiint_S \vec{F} \cdot \vec{n} \, dS = \iiint_Q \operatorname{div} \vec{F} \, dV$$

Closed surface integral
(S is closed)

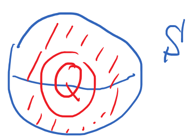
" Outward flux of vector field $\vec{F}$ across a closed surface S is the triple integral of $\operatorname{div} \vec{F}$ "

Compare to 2D Divergence Theorem

$$\oint_C \vec{F} \cdot \vec{n} \, \underline{\underline{ds}} = \iint_R \operatorname{div} \vec{F} \, dA$$



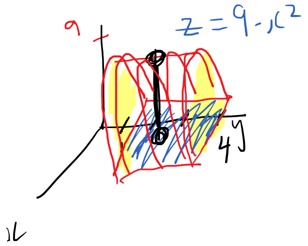
3D $\quad \oiint_S \vec{F} \cdot \vec{n} \, dS = \iiint_Q \operatorname{div} \vec{F} \, dV$



Ex: Compute the (outward) flux of

$\vec{F} = [\,2xy,\ y^2+z,\ y^3\,]$ across

S: bounded by $z=0$, $z=9-x^2$, $y=1$, $y=4$



Note: Want the total flux across all 4 "sides"

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(y^2+z) + \frac{\partial}{\partial z}(y^3)$$
$$= 2y + 2y + 0$$
$$= 4y$$

$$\boxed{Q: \quad \begin{array}{c} 0 \le z \le 9-x^2 \\ 1 \le y \le 4 \\ -3 \le x \le 3 \end{array}}$$

$$\Phi = \iiint_Q \operatorname{div}\vec{F}\, dV$$
$$= \int_{-3}^{3}\int_{1}^{4}\int_{0}^{9-x^2} 4y \; dz\, dy\, dx$$

$$= \int_{-3}^{3}\int_{1}^{4} 4y(9-x^2) \; dy \; dx$$

$$= \int_{-3}^{3} 2y^2(9-x^2)\Big|_{y=1}^{y=4} \, dx$$

$$= \int_{-3}^{3} 30(9-x^2) \, dx$$

$$= 30\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= 30\left[18 - (-18)\right]$$

$$= 1080$$