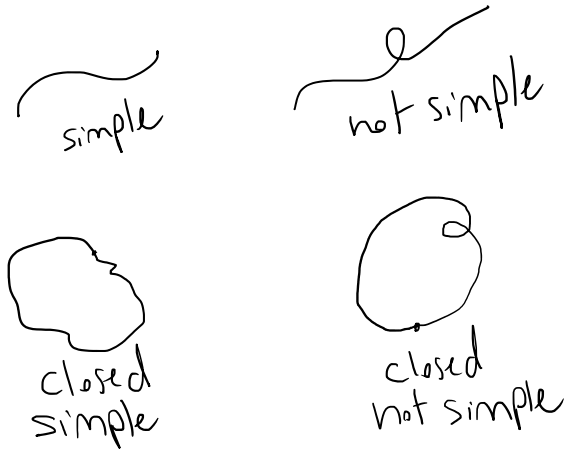
14.4 Green's Theorem

Def

A curve is <u>simple</u> if it has no loops
" <u>closed</u> if start = end



Def

A simple closed curve is <u>positively-oriented</u> if R remains on the left as we travel along C

positively-oriented

<u>Green's Theorem</u> (2D)

Let C be a simple, closed, positively-oriented curve that bounds a region R. Then

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

closed line integral
(C is closed)

Physical Interpretation:

Say $\vec{F} = [P, Q]$ is a force field

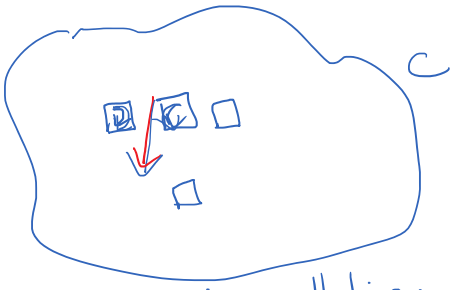
Work done by $\vec{F}$ around $C = \sum_{\substack{\infty\text{-many} \\ \text{tiny rectangles} \\ dA}} \left(z\text{-component of } \operatorname{curl} \vec{F}\right) dA$

Units of $\vec{F}$: N

W : N·m

$\operatorname{curl} \vec{F}$: $\frac{N}{m}$

dA : m^2

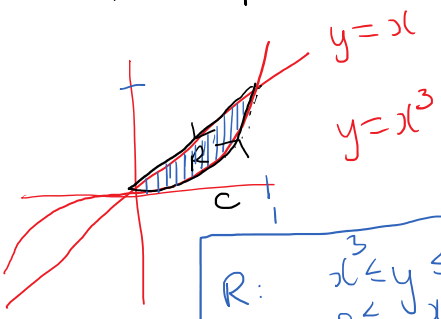


lots of cancellation in interior

$\Rightarrow$ net effect on C

Ex: Evaluate $\oint_C \left[2y^3 dx + (2x^3 + 6xy^2) dy\right]$

C: positively-oriented curve bounded by $y = x$ and $y = x^3$ in the first quadrant



$$\boxed{\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{\partial}{\partial x}\left(2x^3 + 6xy^2\right) - \frac{\partial}{\partial y}\left(2y^3\right) \\ &= 6x^2 + 6y^2 - 6y^2 \\ &= 6x^2 \end{aligned}}$$

$$\boxed{\begin{aligned} R: \quad & x^3 \le y \le x \\ & 0 \le x \le 1 \\ & dA = dy\,dx \end{aligned}}$$

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \int_{x^3}^{x} 6x^2 \, dy\, dx$$

$$= \int_0^1 6x^2 y \Big|_{y=x^3}^{y=x} dx$$

$$= 6\int_0^1 (x^3 - x^5) \quad dx$$

$$= 6 \left[\frac{x^4}{4} - \frac{x^6}{6} \right]_0^1$$

$$= 6 \left(\frac{2}{24} \right)$$

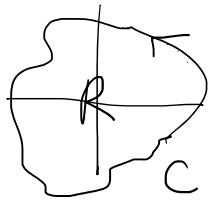
$$= \frac{1}{2}$$

2D

Corollary to Green's Theorem

Let C be a simple, closed, positively-oriented curve that bounds a region R. Then

$$\text{area of } R = \oint_C x\,dy$$



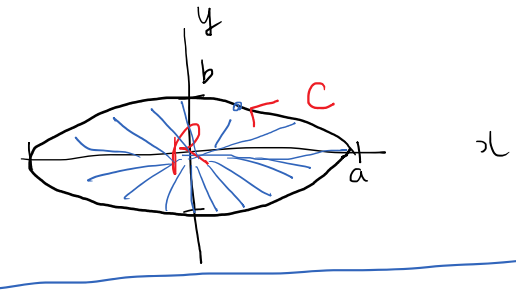
(C is closed)
"closed line integral"

Why? $\oint_C x\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ by Green's Thm.

$= \iint_R 1\, dA$

$=$ area of R

Ex: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

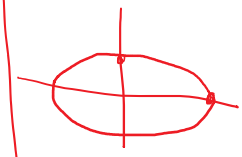


Recall: Elliptical Change of Variables

To parametrize region inside $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$x = ar\cos\theta \quad y = br\sin\theta \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

To parametrize C (ellipse):



$x = a\cos t \qquad y = b\sin t \qquad 0 \le t \le 2\pi$

$dy = b\cos t\, dt$

$$\text{area of ellipse} = \oint_C x\,dy$$

$$= \int_0^{2\pi} a\cos t \cdot b\cos t\, dt$$

$$= ab \int_0^{2\pi} \underbrace{\cancel{\cos^2 t}}_{\frac{1+\cos 2t}{2}}\, dt$$

$$= \frac{ab}{2} \int_0^{2\pi} (1+\cos 2t)\, dt$$

$$= \frac{ab}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$= \frac{ab}{2}(2\pi)$$

$$= \pi ab$$

$$\text{Closed line integral} \xrightarrow{\text{Green's}} \text{Double integral}$$

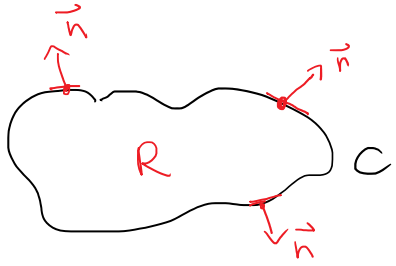
$$\text{Closed line integral} \xleftarrow[\text{Corollary to Green's Thm}]{} \text{Area}$$

Let $\vec{F}$ = fluid flow in 2D

Def

Flux of $\vec{F}$ across curve C = total outflow across C
− total inflow across C

= total net outflow across C

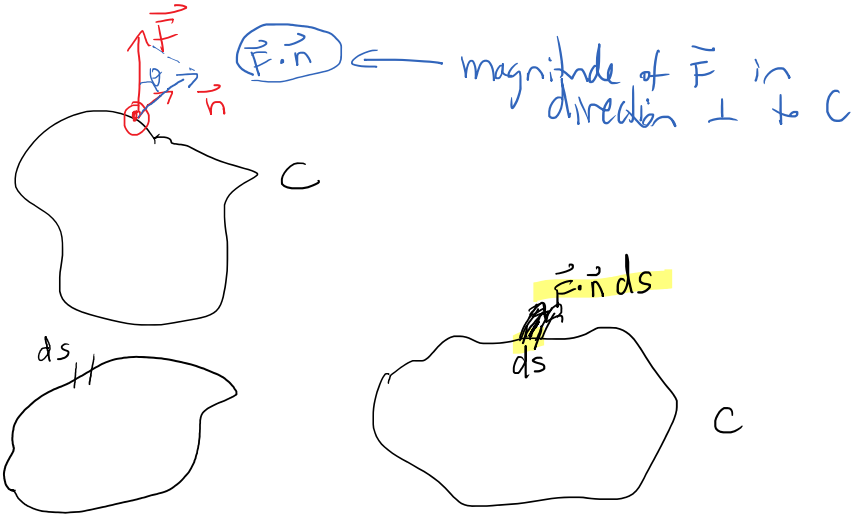


$\vec{n}$ = outer unit normal vector to C

Def

$$\boxed{\text{2D Flux} \quad \Phi = \oint_C \vec{F}\cdot\vec{n}\, ds}$$

total net outflow across C



← magnitude of $\vec{F}$ in direction $\perp$ to C

$$\Phi = \sum \vec{F}\cdot\vec{n}\, ds$$

∞-many tiny chunks ds

To calculate flux:

2D Divergence Theorem (also known as Vector Form of Green's Theorem)

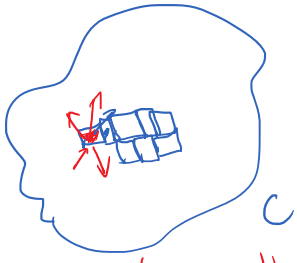
Let C be a simple and closed curve that bounds a region R. Then:

$$\cancel{\oint_C \vec{F}\cdot\vec{n}\,ds} \;\underline{\Phi} = \iint_R (\operatorname{div}\vec{F})\,dA$$

Physical interpretation:

$$\underline{\Phi} = \sum (\operatorname{div}\vec{F})\,dA$$

∞-many tiny rectangles dA



lots of cancellation in interior

$\Rightarrow$ net effect on C (flux of $\vec{F}$ across C)

Units of $\vec{F} = \rho\cdot\vec{v}$: $\frac{kg}{m^2}\cdot\frac{m}{s} = \frac{kg}{m\cdot s}$

ds: m

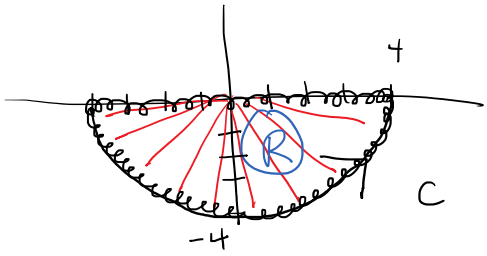
$\operatorname{div}\vec{F}$: $\frac{kg}{m^2\cdot s}$

dA: m^2

$\underline{\Phi}$: $\frac{kg}{s}$

Ex: Calculate the flux of

$\vec{F} = [8xy^2 - 7x,\ 8x^2y + 7y]$ across C:



$$\boxed{\begin{array}{l} 0 \le r \le 4 \\ \pi \le \theta \le 2\pi \\ dA = r\,dr\,d\theta \end{array}}$$

$$\Phi = \iint_R (\operatorname{div} \vec{F})\, dA$$

$$\boxed{\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(8xy^2 - 7x) + \frac{\partial}{\partial y}(8x^2y + 7y) \\ &= 8y^2 - 7 + 8x^2 + 7 \\ &= 8(x^2+y^2) \qquad \rightarrow \text{polar} \\ &= 8r^2 \end{aligned}}$$

$$\Phi = \iint_R (\operatorname{div} \vec{F})\, dA$$

$$= \int_{\pi}^{2\pi}\int_0^4 \cancel{8r^2}\, r\,dr\,d\theta \qquad 8r^3$$

$$= \int_{\pi}^{2\pi} \left[2r^4\right]_{r=0}^{r=4} d\theta$$

$$= 2(4)^4 \cancel{\int_{\pi}^{2\pi} d\theta} \quad \pi$$

$$= 512\pi$$

(if $\vec{F}$ = fluid flow) $\quad \frac{kg}{s}$