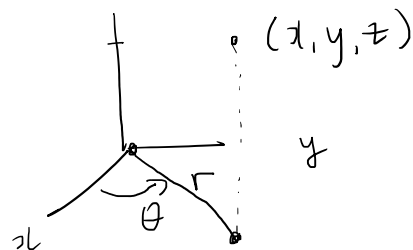


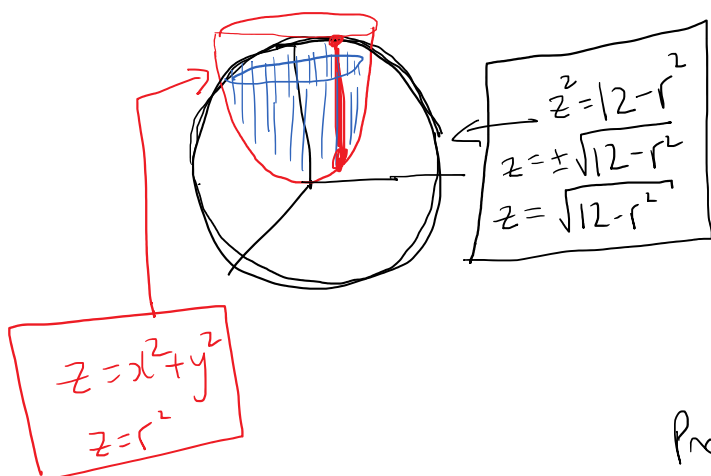
13.7 Integration in Cylindrical and Spherical Coordinates

Cylindrical (r, θ, z)



$$dV = r \, dz \, dr \, d\theta$$

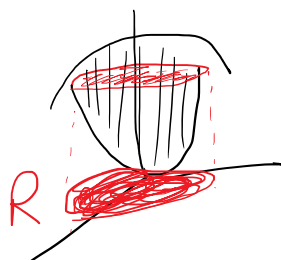
Ex: Set up a triple integral in cylindrical coordinates for the volume bounded by $x^2 + y^2 + z^2 = 12$ and $z = x^2 + y^2$



Slice in z -direction

$$r^2 \leq z \leq \sqrt{12 - r^2}$$

Project on xy -plane:



$R:$

$$z = z$$

$$r^2 = \sqrt{12 - r^2}$$

$$r^4 = 12 - r^2$$

$$r^4 + r^2 - 12 = 0$$

$$(r^2 + 4)(r^2 - 3) = 0$$

no real sol.

$$r^2 = 3$$

$$r = \pm\sqrt{3} \quad r = \sqrt{3}$$

no real sol.

$$r^2 = 3$$

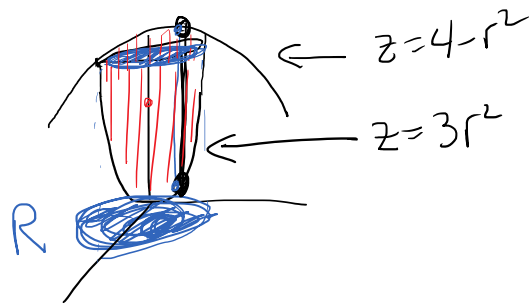
$$r = \pm\sqrt{3} \quad r = \sqrt{3}$$

$$R: \quad 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi$$

$$dV = r \, dz \, dr \, d\theta$$

$$V = \iiint dV \\ = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} r \, dz \, dr \, d\theta$$

Ex. The solid bounded by $z = 3(x^2 + y^2)$ and $z = 4 - x^2 - y^2$ has density = 1 and mass = 2π . Find $\bar{z}$.



Cylindrical

$$3r^2 \leq z \leq 4 - r^2$$

Project on xy -plane

$$z = z$$

$$3r^2 = 4 - r^2$$

$$4r^2 = 4$$

$$r^2 = 1$$

$$r = \pm 1 \quad r = 1$$

$$R: \quad \begin{aligned} 0 &\leq r \leq 1 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

$$dV = r \, dz \, dr \, d\theta$$

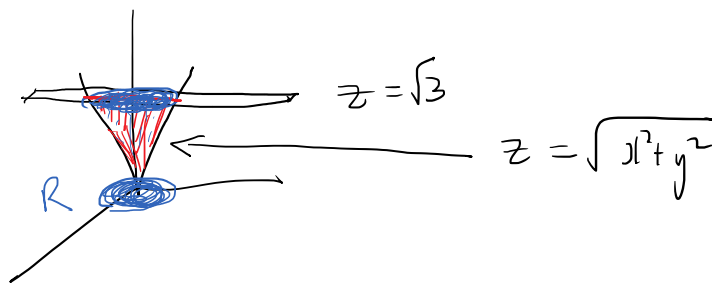
$$\bar{z} = \frac{1}{m} \iiint z \, \delta \, dV$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r^2} z \, r \, dz \, dr \, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \left[\frac{z^2}{2} r \right]_{z=3r^2}^{z=4-r^2} dr \, d\theta$$

$$\begin{aligned}
&= 4\pi \int_0^1 \int_0^1 z - 3r^2 \, dr d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 r \left[\cancel{(4-r^2)^2} - (3r^2)^2 \right] dr d\theta \\
&\quad \quad \quad 16 - 8r^2 + \cancel{r^4} - \cancel{9r^4} - 9r^4 \\
&= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 (16r - 8r^3 - 8r^5) \, dr d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \left(8r^2 - 2r^4 - \frac{8r^6}{\cancel{6}} \right) \bigg|_{r=0}^{r=1} d\theta \\
&\quad \quad \quad \frac{4}{3} \\
&= \frac{1}{4\pi} \int_0^{2\pi} \left[\cancel{8} \frac{2}{6} - \frac{4}{3} \right] d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \frac{14}{3} d\theta \\
&= \frac{1}{4\pi} \left(\frac{14}{3} \right) \int_0^{2\pi} d\theta \\
&\quad \quad \quad \cancel{2\pi} \\
&= \frac{7}{3}
\end{aligned}$$

Ex: Consider the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{3}$. It has density $= \frac{1}{2}$.
Set up a triple integral for $I_x + I_y + I_z$



$$I_x = \iiint \underbrace{(y^2 + z^2)}_{\text{distance from x-axis}} \delta \, dV$$

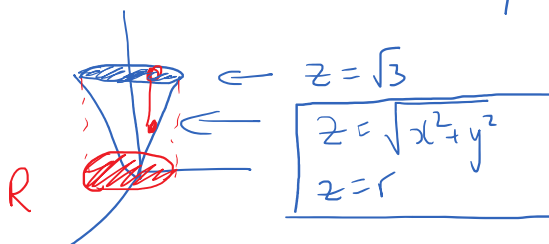
$$I_y = \iiint \underbrace{(x^2 + z^2)}_{\text{distance from y-axis}} \delta \, dV$$

$$I_z = \iiint \underbrace{(x^2 + y^2)}_{\text{distance from z-axis}} \delta \, dV$$

$$I_x + I_y + I_z = \iiint \frac{(2x^2 + 2y^2 + 2z^2)}{2(x^2 + y^2 + z^2)} \delta \, dV$$

$$= \iiint (x^2 + y^2 + z^2) \, dV$$

Cylindrical



Slice in z -direction
 $r \leq z \leq \sqrt{3}$

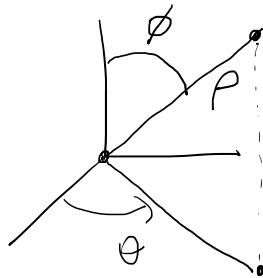
$$R: \quad \begin{aligned} z &= z \\ r &= \sqrt{3} \end{aligned}$$

$$R: \quad 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi$$

$$dV = r \, dz \, dr \, d\theta$$

$$\begin{aligned} \bar{I}_x + \bar{I}_y + \bar{I}_z &= \iiint (x^2 + y^2 + z^2) \, dV \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{\sqrt{3}} (r^2 + z^2) \, r \, dz \, dr \, d\theta \end{aligned}$$

Spherical coordinates (ρ, ϕ, θ)



$$dV = \underbrace{\rho^2 \sin \phi}_{\text{Jacobian}} d\rho d\phi d\theta$$

Ex. Set up a triple integral in spherical coordinates for the mass of the solid bounded by $z = \sqrt{3} \sqrt{x^2 + y^2}$ and $z = 3$ if $\delta = k$



$$z = 3$$

$$z = \sqrt{3} \sqrt{x^2 + y^2}$$

$$\begin{aligned} z &= 3 \\ \rho \cos \phi &= 3 \\ \rho &= 3 \sec \phi \\ \underline{0 \leq \rho \leq 3 \sec \phi} \end{aligned}$$

$$\begin{aligned} z &= \sqrt{3} \sqrt{x^2 + y^2} \\ z &= \sqrt{3} r \\ \rho \cos \phi &= \sqrt{3} \rho \sin \phi \\ 1 &= \sqrt{3} \tan \phi \\ \tan \phi &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\tan \phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

$$0 \leq \phi \leq \frac{\pi}{6}$$

$$0 \leq \theta \leq 2\pi$$

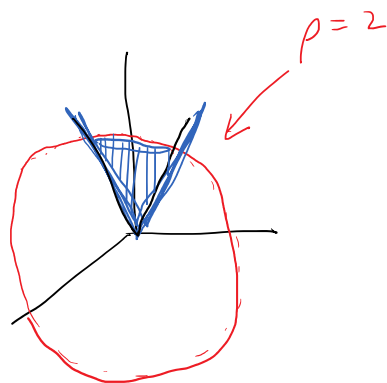
$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$m = \iiint \delta \, dV$$

$$= k \iiint \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= k \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{3\sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Ex. Find the volume inside $\rho = 2$ and above $z = 1$.

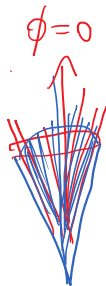


$$z = r$$

$$z = \sqrt{x^2 + y^2}$$

Spherical

$$0 \leq \rho \leq 2$$



$$\begin{aligned}
 z &= r \\
 \rho \cos \phi &= \rho \sin \phi \\
 \cos \phi &= \sin \phi \\
 1 &= \tan \phi \\
 \phi &= \frac{\pi}{4} \\
 0 \leq \phi &\leq \frac{\pi}{4}
 \end{aligned}$$

$$0 \leq \theta \leq 2\pi$$

$$dV = \underbrace{\rho^2 \sin \phi}_{\text{volume element}} d\rho d\phi d\theta$$

$$V = \iiint dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin \phi \right]_{\rho=0}^{\rho=2} d\phi d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin \phi d\phi d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \left[-\cos \phi \right]_{\phi=0}^{\phi=\pi/4} d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \left[-\frac{\sqrt{2}}{2} + 1 \right] d\theta$$

$$= \frac{8}{3} \cdot \frac{(2-\sqrt{2})}{2} \int_0^{2\pi} d\theta$$

$$= \frac{8(2-\sqrt{2})\pi}{3}$$