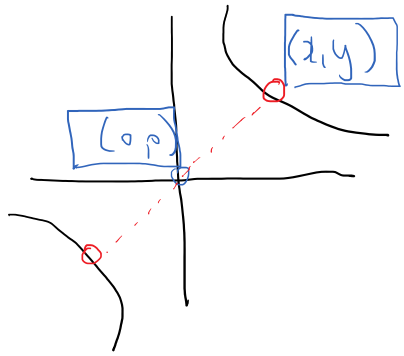
12.9 Lagrange Multipliers

Goal: Maximize or minimize a function f subject to a constraint $g = \text{constant}$

$$\boxed{\text{Method: Let } \nabla f = \lambda \nabla g}$$

Ex: Find the points on $xy = 6$ that are closest to the origin.



$xy = 6$

Let the points be (x,y)

Minimize distance $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(x-0)^2 + (y-0)^2}$$

$$= \sqrt{x^2 + y^2}$$

Equivalent: Minimize $f = x^2 + y^2$

(yields the same solutions)

Constraint: $\underbrace{xy}_{g} = 6$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y] = \lambda [y, x]$$

$$\boxed{\begin{array}{ll} (1)\quad 2x = \lambda y & (3)\quad xy = 6 \\ (2)\quad 2y = \lambda x & \end{array}}$$

(1): $\lambda = \frac{2x}{y}$

(2): $\lambda = \frac{2y}{x}$

$$\lambda = \lambda$$

$$\frac{2x}{y} = \frac{2y}{x}$$

$$2x^2 = 2y^2$$

$$y^2 = x^2$$

$$y = \pm x$$

(3): $xy = 6 \Rightarrow x$ and y have same sign

$\Rightarrow \boxed{y = x}$

$y = x \rightarrow (3)$: $x^2 = 6$

$x = \pm\sqrt{6}$

$x = \sqrt{6}$ $\quad y = x = \sqrt{6}$

$x = -\sqrt{6}$ $\quad y = x = -\sqrt{6}$

Points are $(x, y) = \pm(\sqrt{6}, \sqrt{6})$

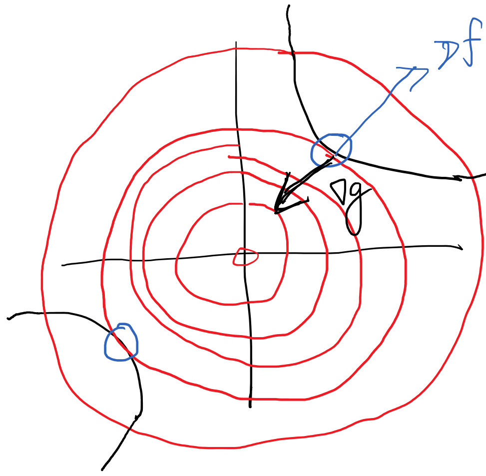
Lagrange Multiplier method selects the level curve of f that's <u>tangent</u> to the constraint.

$f = x^2 + y^2$

Level curves of f

$x^2 + y^2 = \text{constant}$

(circles)



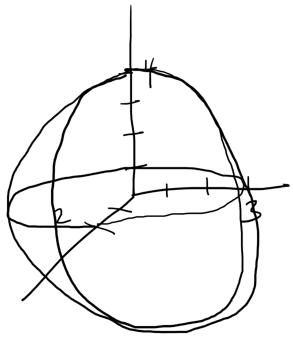
Constraint

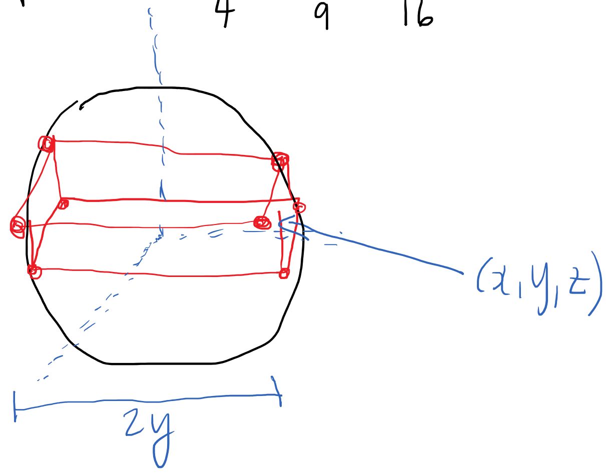
$xy = 6$

(hyperbola)

$$\nabla f = \lambda \nabla g$$

Ex: Find the maximum volume of a rectangular box inscribed inside the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$





$$V=(2x)(2y)(2z)$$

Maximize $f=8xyz$ Constraint $\underbrace{\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}}_{g}=1$

$$\nabla f=\lambda \nabla g$$

$$[8yz,\ 8xz,\ 8xy]=\lambda\left[\frac{2x}{4},\ \frac{2y}{9},\ \frac{2z}{16}\right]$$

$$\begin{cases} ① & 8yz=\lambda\left(\frac{2x}{4}\right) \longrightarrow \lambda=8yz\left(\frac{4}{2x}\right)=\frac{16yz}{x} \\ ② & 8xz=\lambda\left(\frac{2y}{9}\right) \rightarrow \lambda=8xz\left(\frac{9}{2y}\right)=\frac{36xz}{y} \\ ③ & 8xy=\lambda\left(\frac{2z}{16}\right) \rightarrow \lambda=8xy\left(\frac{16}{2z}\right)=\frac{64xy}{z} \end{cases}$$

$$\boxed{④\ \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1}$$

$$\lambda = \lambda = \lambda$$

$$\boxed{\frac{16yz}{x} = \frac{36xz}{y} = \frac{64xy}{z}}$$

Get y and z in terms of x

$$\frac{16yz}{x} = \frac{36xz}{y}$$

$(z \neq 0)$

$$\frac{16y}{x} = \frac{36x}{y}$$

$$16y^2 = 36x^2$$

$$y^2 = \frac{36x^2}{16}$$

$$\frac{16yz}{x} = \frac{64xy}{z}$$

$(y \neq 0)$

$$\frac{16z}{x} = \frac{64x}{z}$$

$$16z^2 = 64x^2$$

$$z^2 = \frac{64x^2}{16}$$

(4) :

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

$$\frac{x^2}{4} + \frac{1}{\cancel{9}}\left(\frac{\cancel{36}^{\,4}x^2}{\cancel{16}_{\,4}}\right) + \frac{1}{\cancel{16}}\left(\frac{\cancel{64}^{\,4}x^2}{\cancel{16}_{\,4}}\right) = 1$$

$$\frac{3x^2}{4} = 1$$

$$x^2 = \frac{4}{3}$$

$$x = \pm\sqrt{\frac{4}{3}} \qquad (x > 0)$$

$$\boxed{x = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}}$$

$$y^2 = \frac{36x^2}{16}$$

$$= \frac{36}{16}\left(\frac{4}{3}\right)$$

$$= 3$$

$$y = \pm\sqrt{3} \quad (y > 0)$$

$$\boxed{y = \sqrt{3}}$$

$$z^2 = \frac{\overset{4}{\cancel{64}}x^2}{\cancel{16}}$$

$$= 4\left(\frac{4}{3}\right)$$

$$= \frac{16}{3}$$

$$z = \pm\frac{4}{\sqrt{3}} \quad (z > 0)$$

$$\boxed{z = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}}$$

$$\text{Max volume} = 8xyz = 8\left(\frac{2\cancel{\sqrt{3}}}{\cancel{3}}\right)\left(\sqrt{3}\right)\left(\frac{4\cancel{\sqrt{3}}}{3}\right) = \frac{64\sqrt{3}}{3},$$

achieved when $(x, y, z) = \left(\frac{2\sqrt{3}}{3}, \sqrt{3}, \frac{4\sqrt{3}}{3}\right)$