12.7 Multivariable Chain Rule

Multivariable Chain Rule

Single-Variable Chain Rule

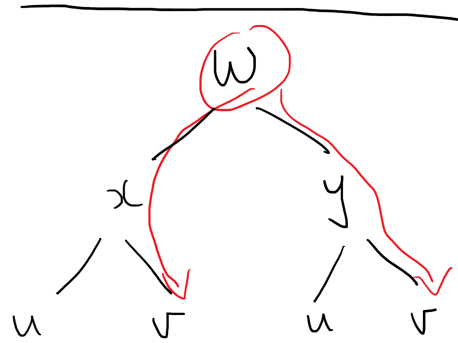
y depends on x

and x " t

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Multivariable Chain Rule

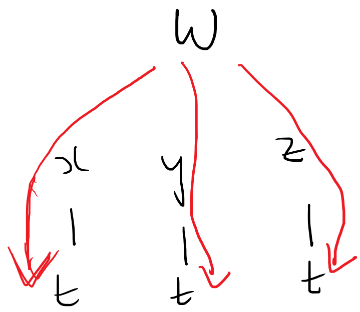
Ex:



Find $\frac{\partial w}{\partial v}$

$$\boxed{\frac{\partial w}{\partial v}} = \boxed{\frac{\partial w}{\partial x}} \cdot \boxed{\frac{\partial x}{\partial v}} + \boxed{\frac{\partial w}{\partial y}} \cdot \boxed{\frac{\partial y}{\partial v}}$$

Ex:



Find $\dfrac{dw}{dt}$

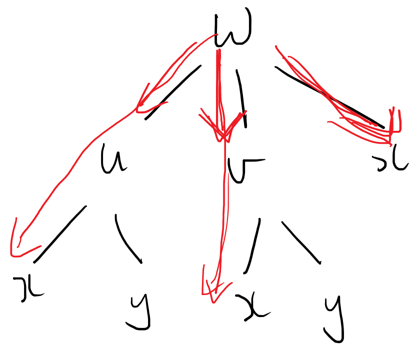
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

derivative

partial derivative

derivative

Ex:



f

Find $\dfrac{\partial w}{\partial x}$

Caution: x appears in multiple levels

To avoid confusion, say $\boxed{w = f(u, v, x)}$

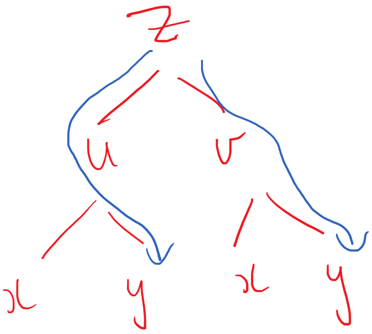
$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}$$

Ex: $z = f(u,v)$

$u = 3x + y^3 \qquad v = xy - 3y^2$

$z_u = -3$ and $z_v = 4$ at $(u,v) = (11,-10)$

Find z_y at $(x,y) = (1,2)$



$$z_y = z_u \boxed{u_y} + z_v \boxed{v_y}$$

$$z_y = z_u(3y^2) + z_v(x - 6y)$$

$$\boxed{\begin{array}{l} (x,y) = (1,2) \Rightarrow \quad u = 3x + y^3 = 11 \qquad v = xy - 3y^2 = -10 \\ \qquad \Rightarrow \quad z_u = -3 \qquad z_v = 4 \end{array}}$$

$$z_y = -3(12) + 4(-11)$$

$$= -80$$

Multivariable implicit functions

e.g. $x^2 + y^2 + z^2 + 6xyz^3 - 2x + y = 0$

Context

- z depends on x and y
- x and y are independent

Warm Up

$$\boxed{\begin{array}{ll} \frac{\partial}{\partial x}[x^2] = 2x & \frac{\partial}{\partial y}[x^2] = 0 \\ \frac{\partial}{\partial x}[y^2] = 0 & \frac{\partial}{\partial y}[y^2] = 2y \\ \frac{\partial}{\partial x}[z^2] = 2z\frac{\partial z}{\partial x} & \frac{\partial}{\partial y}[z^2] = 2z\frac{\partial z}{\partial y} \end{array}}$$

Chain Rule

Ex. Find $\frac{\partial z}{\partial y}$ given

$y(6xz^3)$

$$\boxed{x^2 + y^2 + z^2 + 6xyz^3 - 2x + y = 0}$$

Take $\frac{\partial}{\partial y}$:

$$0 + 2y + 2z\frac{\partial z}{\partial y} + \underbrace{y\left(18xz^2\frac{\partial z}{\partial y}\right) + 6xz^3}_{\text{Product Rule}} + 0 + 1 = 0$$

$$2z\frac{\partial z}{\partial y} + 18xyz^2\frac{\partial z}{\partial y} = -2y - 6xz^3 - 1$$

$$\left[2z + 18xyz^2\right]\frac{\partial z}{\partial y} = -1 - 2y - 6xz^3$$

$$\frac{\partial z}{\partial y} = \frac{-\left(1 + 2y + 6xz^3\right)}{2z + 18xyz^2}$$