1

1) $\vec{X}_c$:

$$\begin{vmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-2-\lambda)+6=0$$

$$\lambda^2-\lambda=0$$

$$\lambda(\lambda-1)=0$$

$$\lambda=0,1$$

$\lambda=0$:

$$[A-0I\,|\,\vec{0}]$$

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$\frac{R_1}{3} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2-2R_1 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_2=a$$

$$x_1-x_2=0 \Rightarrow x_1=a$$

$$\vec{x}=\begin{bmatrix} 1 \\ 1 \end{bmatrix} a \qquad \vec{K}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda=1$:

$$[A-I\,|\,\vec{0}]$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right]$$

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 2 & -3 & 0 \end{array}\right] \rightarrow$$

① Cont'd

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_2 = a$$

$$x_1 - 3/2\, x_2 = 0 \implies x_1 = 3/2\, a$$

$$\vec{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} a \text{ or } \begin{bmatrix} 3 \\ 2 \end{bmatrix} a \qquad \vec{K}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{X}_c = c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{X}_1} + c_2 \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t}_{\vec{X}_2}$$

$$\Phi = \begin{bmatrix} 1 & 3e^t \\ 1 & 2e^t \end{bmatrix}$$

2) Φ^{-1}

$$|\Phi| = 2e^t - 3e^t$$
$$= -e^t$$

$$\Phi^{-1} = \frac{1}{-e^t} \begin{bmatrix} 2e^t & -3e^t \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{bmatrix}$$

① Cont'd

3) $\vec{u} = \int \Phi^{-1}\vec{F}\,dt$

$$= \int \begin{bmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} dt$$

$$= \int \begin{bmatrix} -11 \\ 5e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -11t \\ -5e^{-t} \end{bmatrix}$$ Don't use constants here.

4) $\vec{X}_p = \Phi\,\vec{u}$

$$= \begin{bmatrix} 1 & 3e^{t} \\ 1 & 2e^{t} \end{bmatrix}\begin{bmatrix} -11t \\ -5e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -11t-15 \\ -11t-10 \end{bmatrix}$$

5) $\vec{X} = \vec{X}_c + \vec{X}_p$

$$= C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{t} + \begin{bmatrix} -11t-15 \\ -11t-10 \end{bmatrix}$$

or $\vec{X} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{t} - \begin{bmatrix} 11 \\ 11 \end{bmatrix}t - \begin{bmatrix} 15 \\ 10 \end{bmatrix}$

2

Section 8.3

1) $\vec{X}_c$

$$\begin{vmatrix} 3-\lambda & -5 \\ 0.75 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + \frac{15}{4} = 0$$

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0$$

$$4\lambda^2 - 8\lambda + 3 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-48}}{8}$$

$$\lambda = \frac{8 \pm \sqrt{16}}{8}$$

$$\lambda = \frac{8 \pm 4}{8}$$

$$\lambda = \frac{1}{2}, \frac{3}{2}$$

$\lambda = \frac{1}{2}$:

$$[A - \tfrac{1}{2} I \,|\, \vec{0}\,]$$

$$\left[\begin{array}{cc|c} 5/2 & -5 & 0 \\ 3/4 & -3/2 & 0 \end{array}\right]$$

$\frac{2}{5} R_1$ $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3/4 & -3/2 & 0 \end{array}\right]$$

$R_2 - \frac{3}{4} R_1$ $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2a$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} a \qquad \vec{K}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

② Cont'd

$\lambda = \frac{3}{2}$:

$$[A - \tfrac{3}{2} I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 3/2 & -5 & 0 \\ 3/4 & -5/2 & 0 \end{array}\right]$$

$$\tfrac{2}{3}R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{10}{3} & 0 \\ 3/4 & -5/2 & 0 \end{array}\right]$$

$$R_2 - \tfrac{3}{4}R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{10}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_2 = a$$

$$x_1 - \tfrac{10}{3}x_2 = 0 \Rightarrow x_1 = \tfrac{10}{3}a$$

$$\vec{x} = \begin{bmatrix} 10/3 \\ 1 \end{bmatrix} a \text{ or } \begin{bmatrix} 10 \\ 3 \end{bmatrix} a \quad \vec{K}_2 = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\vec{X}_c = c_1 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t/2}}_{\vec{X}_1} + c_2 \underbrace{\begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{3t/2}}_{\vec{X}_2}$$

$$\Phi = \begin{bmatrix} 2e^{t/2} & 10e^{3t/2} \\ e^{t/2} & 3e^{3t/2} \end{bmatrix}$$

2) Φ^{-1}

$$|\Phi| = 6e^{2t} - 10e^{2t}$$
$$= -4e^{2t} \quad \rightarrow$$

$$\Phi^{-1} = \frac{1}{-4e^{2t}} \begin{bmatrix} 3e^{3t/2} & -10e^{3t/2} \\ -e^{t/2} & 2e^{t/2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{4}e^{-t/2} & \frac{5}{2}e^{-t/2} \\ \frac{1}{4}e^{-3t/2} & -\frac{1}{2}e^{-3t/2} \end{bmatrix} \text{ or } \frac{1}{4}\begin{bmatrix} -3e^{-t/2} & 10e^{-t/2} \\ e^{-3t/2} & -2e^{-3t/2} \end{bmatrix}$$

3) $$\bar{u} = \int \Phi^{-1} \vec{F}\, dt$$

$$= \frac{1}{4}\int \begin{bmatrix} -3e^{-t/2} & 10e^{-t/2} \\ e^{-3t/2} & -2e^{-3t/2} \end{bmatrix} \begin{bmatrix} e^{t/2} \\ -e^{t/2} \end{bmatrix} dt$$

$$= \frac{1}{4}\int \begin{bmatrix} -13 \\ 3e^{-t} \end{bmatrix} dt$$

$$= \frac{1}{4}\begin{bmatrix} -13t \\ -3e^{-t} \end{bmatrix}$$ Don't use constants here.

4) $$\bar{X}_p = \Phi \bar{u}$$

$$= \frac{1}{4}\begin{bmatrix} 2e^{t/2} & 10e^{3t/2} \\ e^{t/2} & 3e^{3t/2} \end{bmatrix} \begin{bmatrix} -13t \\ -3e^{-t} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -26te^{t/2} - 30e^{t/2} \\ -13te^{t/2} - 9e^{t/2} \end{bmatrix}$$

or $$\begin{bmatrix} -6.5te^{t/2} - 7.5e^{t/2} \\ -3.25te^{t/2} - 2.25e^{t/2} \end{bmatrix}$$

5) $$\vec{X} = C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t/2} + C_2\begin{bmatrix} 10 \\ 3 \end{bmatrix}e^{3t/2} - \begin{bmatrix} 6.5 \\ 3.25 \end{bmatrix}te^{t/2} - \begin{bmatrix} 7.5 \\ 2.25 \end{bmatrix}e^{t/2}$$

3 1) $\vec{X}_c$ Section 8.3

$$\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda(3-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, 2$$

$\lambda = 1$: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$\frac{R_1}{-1} \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 + R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2a$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} a \qquad \vec{K}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 2$: $[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\frac{R_1}{-2} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 + R_1 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 - x_2 = 0 \Rightarrow x_1 = a \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a \qquad \vec{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

③ Cont'd

$$\bar{X}_c = c_1 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t}_{\vec{X}_1} + c_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}}_{\vec{X}_2}$$

$$\Phi = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

2) Φ^{-1}

$$|\Phi| = 2e^{3t} - e^{3t}$$
$$= e^{3t}$$

$$\Phi^{-1} = \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

3) $$\bar{u} = \int \Phi^{-1} \vec{F} \, dt$$

$$= \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt$$

$$= \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$ Don't use constants here.

4) $$\vec{X}_p = \Phi \vec{u}$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$\rightarrow$

③ Cont'd

Section 8.3

$$\vec{X}_p = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 4te^{t} + 3e^{t} \\ 2te^{t} + 3e^{t} \end{bmatrix}$$

5) $\bar{X} = \bar{X}_c + \vec{X}_p$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 4te^{t} + 3e^{t} \\ 2te^{t} + 3e^{t} \end{bmatrix}$$

or $\bar{X} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^{t}$

4) 1) $\bar{X}_c$

Section 8.3

$$\begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 8 = 0$$

$$\lambda^2 - 9 = 0$$

$$(\lambda-3)(\lambda+3) = 0$$

$$\lambda = 3, -3$$

$\lambda = 3$: $[A - 3I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} -2 & 8 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$\frac{R_1}{-2} \quad \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4a$

$\bar{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} a \quad \bar{K}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda = -3$: $[A + 3I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} 4 & 8 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$\frac{R_1}{4} \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2a \quad \bar{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} a \quad \bar{K}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(4) Cont'd

$$\bar{X}_c = c_1 \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t}}_{\bar{X}_1} + c_2 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t}}_{\bar{X}_2}$$

$$\Phi = \begin{bmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix}$$

2) Φ^{-1}

$$|\Phi| = 4 + 2$$
$$= 6$$

$$\Phi^{-1} = \frac{1}{6} \begin{bmatrix} e^{-3t} & 2e^{-3t} \\ -e^{3t} & 4e^{3t} \end{bmatrix}$$

3) $\bar{u} = \int \Phi^{-1} \bar{F} \, dt$

$$= \frac{1}{6} \int \begin{bmatrix} e^{-3t} & 2e^{-3t} \\ -e^{3t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} 12t \\ 12t \end{bmatrix} dt$$

$$= \frac{1}{6} \int \begin{bmatrix} 36te^{-3t} \\ 36te^{3t} \end{bmatrix} dt$$

$$= \int \begin{bmatrix} 6te^{-3t} \\ 6te^{3t} \end{bmatrix} dt$$

$$\bar{u} = \begin{bmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{bmatrix}$$

Don't use constants here.

	D	I
+	$6t$	e^{-3t}
−	6	$-\frac{1}{3}e^{-3t}$
		$\frac{1}{9}e^{-3t}$

	D	I
+	$6t$	e^{3t}
−	6	$\frac{1}{3}e^{3t}$
		$\frac{1}{9}e^{3t}$

→

4) $\bar{X}_p = \Phi \bar{u}$

$$= \begin{bmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -2t\,e^{-3t} & -\frac{2}{3}e^{-3t} \\ 2t\,e^{3t} & -\frac{2}{3}e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} -12t & -\frac{4}{3} \\ 0t & -\frac{4}{3} \end{bmatrix}$$

5) $\bar{X} = \bar{X}_c + \bar{X}_p$

$$= C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -12t - \frac{4}{3} \\ -\frac{4}{3} \end{bmatrix}$$

or $\bar{X} = C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} -12 \\ 0 \end{bmatrix} t - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$