(1) $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda)-8=0$

$\lambda^2-4\lambda-5=0$

$(\lambda-5)(\lambda+1)=0$

$\lambda=-1,5$

$\lambda=5$: $[A-5I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$\frac{R_1}{-4}$ $\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \end{array}\right]$

R_2-4R_1 $\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2=a$

$x_1-\frac{1}{2}x_2=0 \Rightarrow x_1=\frac{1}{2}a$

$\vec{x}=\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}a$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}a$

$\vec{K}_1=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda=-1$: $[A+I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

① Cont'd

$\frac{R_1}{2}$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right]$

$R_2 - 4R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\uparrow$

$x_2 = a$

$x_1 + x_2 = 0 \Rightarrow x_1 = -a$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} a$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix} a$

$\vec{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$

$\vec{X} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

②

$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$

$(1-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda)(-1-\lambda) = 0$

$\lambda = 1, 2, -1$

$\rightarrow$

(2) Cont'd

$\lambda = 1$: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 + R_2 \\ \\ R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = a$

$x_2 = 0, \quad x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a$$

$$\vec{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 2$: $[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\frac{R_1}{-1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

② Cont'd

R_1+R_2 $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = a$

$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2a$

$x_2 - 3x_3 = 0 \Rightarrow x_2 = 3a$

$$\bar{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} a$$

$$\bar{K}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$\lambda = -1$: $[A + I \mid \bar{0}]$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$\frac{R_1}{2}$ $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$\frac{R_2}{3}$ $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$R_1 - \frac{1}{2}R_2$
$R_3 - R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = a$

$x_1 - \frac{1}{2}x_3 = 0 \Rightarrow x_1 = \frac{1}{2}a$

$x_2 = 0$

$\bar{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} a$ or $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} a$ $\qquad \bar{K}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

→

(2) Cont'd

$$\bar{X} = \sum C_i \vec{K_i} e^{\lambda_i t}$$

$$\vec{X} = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{-t}$$

(3)

$$\begin{vmatrix} 0.5-\lambda & 0 \\ 1 & -0.5-\lambda \end{vmatrix} = 0$$

$$(0.5-\lambda)(-0.5-\lambda) = 0$$

$\lambda = 0.5$ or $\lambda = \frac{1}{2}$ $\qquad$ $\lambda = -0.5$ or $\lambda = -\frac{1}{2}$

$\lambda = \frac{1}{2}$:

$$[A - \tfrac{1}{2} I \mid \bar{0}]$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$x_1 - x_2 = 0 \Rightarrow x_1 = a$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a \qquad \bar{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -\frac{1}{2}$:

$$[A + \tfrac{1}{2} I \mid \bar{0}]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$

③ Cont'd

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$x_2 = a$

$x_1 = 0$

$\bar{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} a \qquad \bar{K}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{X} = \sum C_i \vec{K_i} e^{\lambda_i t}$$

$$\boxed{\bar{X} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2}}$$

Sub $t=0$

$\bar{X} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$: $\quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$3 = C_1$ ①

$5 = C_1 + C_2$ ②

① → ② : $\quad 5 = 3 + C_2$

$C_2 = 2$

$$\bar{X} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2}$$

(4)

$$\begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda)+9=0$$

$$\lambda^2 = 0$$

$$\lambda \cdot \lambda = 0$$

$$\lambda = 0, 0$$

$\lambda = 0$: $[A \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 9 & -3 & 0 \end{array}\right]$$

$$\frac{R_1}{3} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 9 & -3 & 0 \end{array}\right]$$

$$R_2 - 9R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$x_2 = a$

$$x_1 - \frac{1}{3}x_2 = 0 \Rightarrow x_1 = \frac{1}{3}a$$

$$\vec{x} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} a \text{ or } \begin{bmatrix} 1 \\ 3 \end{bmatrix} a \qquad \vec{K}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

We are in the case of repeated eigenvalues without enough eigenvectors.

Solve $[A - \lambda I]\vec{P} = \vec{K}_1$ for $\vec{P}$ $\rightarrow$

(4) Cont'd

Solve $[A - \lambda I]\vec{P} = \vec{K}_1$ for $\vec{P}$

Recall $\lambda = 0$: $\quad [A]\vec{P} = \vec{K}_1$

$$\begin{array}{cc} p_1 & p_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 9 & -3 & 3 \end{array}\right]$$

$\frac{R_1}{3}$ $\quad \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{1}{3} \\ 9 & -3 & 3 \end{array}\right]$

$R_2 - 9R_1$ $\quad \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right]$

$p_2 = a$

$p_1 - \frac{1}{3}p_2 = \frac{1}{3} \Rightarrow p_1 = \frac{1}{3} + \frac{1}{3}a$

$$\vec{P} = \begin{bmatrix} \frac{1}{3} + \frac{1}{3}a \\ a \end{bmatrix}$$

Choose any nonzero $\vec{P}$: $\quad a = 2 \Rightarrow \vec{P} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\text{There are many other possibilities e.g. } a = 0 \Rightarrow \vec{P} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \text{ etc.}}$

Solution $\quad \vec{X} = c_1 \vec{K}_1 e^{\lambda_1 t} + c_2(\vec{K}_1 t + \vec{P})e^{\lambda_1 t}$

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

(5)

Section 8.2

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1-\lambda \\ 1 & -1 \end{vmatrix} = 0$$

$$(3-\lambda)[(1-\lambda)^2-1] + [1-\lambda+1] - [-1-1+\lambda] = 0$$

$$(3-\lambda)(\lambda^2-2\lambda) + (2-\lambda) + 2-\lambda = 0$$

$$(3-\lambda)\lambda(\lambda-2) - (\lambda-2) - (\lambda-2) = 0$$

Factor!

$$(\lambda-2)[(3-\lambda)\lambda - 1 - 1] = 0$$

$$(\lambda-2)[-\lambda^2+3\lambda-2] = 0$$

$$-(\lambda-2)(\lambda^2-3\lambda+2) = 0$$

$$-(\lambda-2)(\lambda-1)(\lambda-2) = 0$$

$$-(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2$$

$\lambda = 1$:

$$[A - I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \to$$

⑤ Cont'd

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2-2R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\frac{R_2}{-1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_3+R_2 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_3 = a$$

$$x_1 - x_3 = 0 \Rightarrow x_1 = a$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = a$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} a \quad \bar{K}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$[A - 2I \mid \bar{0}]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow \quad \uparrow$$

$$x_2 = a \quad x_3 = b$$

$$x_1 - a - b = 0 \Rightarrow x_1 = a + b$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} b \qquad \bar{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{K}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$

(5) Cont'd

$$\bar{X} = \sum C_i \bar{K}_i e^{\lambda_i t}$$

$$\bar{X} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

(6)

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[-\lambda(2-\lambda)+1\right] = 0$$

$$(1-\lambda)\left[\lambda^2 - 2\lambda + 1\right] = 0$$

$$(1-\lambda)(\lambda-1)^2 = 0$$

$$-(\lambda-1)^3 = 0$$

$$\lambda = 1, 1, 1$$

$\lambda = 1$: $\quad [A - I \mid \bar{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Reorder rows $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\frac{R_1}{2}$ $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$

⑥ Cont'd

$$R_1 - \tfrac{1}{2}R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = a$

$x_1 = 0$

$x_2 - x_3 = 0 \Rightarrow x_2 = a$

$$\bar{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} a \qquad \bar{K}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We are in the case of repeated eigenvalues without enough eigenvectors.

Solve $[A - \lambda I]\vec{P} = \bar{K}_1$ for $\vec{P}$

$$[A - I]\vec{P} = \bar{K}_1$$

$$\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \end{array}$$

Reorder rows $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\frac{R_1}{2}$ $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 - \frac{1}{2}R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$P_3 = a$

$P_1 = 0$

$P_2 \quad - P_3 = 1 \Rightarrow P_2 = 1 + a$

$$\vec{P} = \begin{bmatrix} 0 \\ 1+a \\ a \end{bmatrix}$$

Choose any nonzero $\vec{P}$:

$$\vec{P} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow$$

⑥ Cont'd

Section 8.2

We still need a third vector.

Solve $[A-\lambda I]\vec{Q}=\vec{P}$ for $\vec{Q}$

$$[A-I]\vec{Q}=\vec{P}$$

$$\begin{array}{ccc} q_1 & q_2 & q_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Reorder rows $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\frac{R_1}{2}$ $\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_1-\frac{1}{2}R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\uparrow$

$q_3=a$

$q_1=\frac{1}{2}$

$q_2-q_3=0 \Rightarrow q_2=a$

$$\vec{Q}=\begin{bmatrix}\frac{1}{2}\\ a\\ a\end{bmatrix}$$

Choose any nonzero $\vec{Q}$: $\vec{Q}=\begin{bmatrix}0.5\\0\\0\end{bmatrix}$

$$\vec{X}=C_1\vec{K}_1e^{\lambda_1 t}+C_2\left(\vec{K}_1t+\vec{P}\right)e^{\lambda_1 t}+C_3\left(\vec{K}_1\frac{t^2}{2}+\vec{P}t+\vec{Q}\right)e^{\lambda_1 t}$$

$$\vec{X}=\left[C_1\begin{bmatrix}0\\1\\1\end{bmatrix}+C_2\left(\begin{bmatrix}0\\1\\1\end{bmatrix}t+\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+C_3\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2}{2}+\begin{bmatrix}0\\1\\0\end{bmatrix}t+\begin{bmatrix}0.5\\0\\0\end{bmatrix}\right)\right]e^{t}$$

(7)

$$\begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(2-\lambda)+5=0$$

$$\lambda^2-8\lambda+17=0$$

$$\lambda=\frac{8\pm\sqrt{64-68}}{2}$$

$$\lambda=\frac{8\pm\sqrt{-4}}{2} \leftarrow 2i$$

$$\lambda=\frac{8\pm 2i}{2}$$

$$\lambda=4\pm i$$

Find an eigenvector $\vec{K}_1$ for $\lambda=\alpha+\beta i$

$\lambda=4+i$: $[A-(4+i)I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 5 & -2-i & 0 \\ 2-i & -1 & 0 \end{array}\right]$$

$$\frac{R_1}{5} \quad \left[\begin{array}{cc|c} 1 & \frac{-2-i}{5} & 0 \\ 2-i & -1 & 0 \end{array}\right]$$

$$R_2-(2-i)R_1 \quad \left[\begin{array}{cc|c} 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} &-1-(2-i)\left(\frac{-2-i}{5}\right) \\ &= -1+\frac{(2-i)(2+i)}{5} \\ &= -1+\frac{5}{5} \end{aligned}}$$

7 Cont'd

$$\left[\begin{array}{cc|c} 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$

$$x_1 - \left(\frac{2+i}{5}\right)x_2 = 0 \implies x_1 = \frac{2+i}{5}a$$

$$\bar{x} = \begin{bmatrix} \frac{2+i}{5} \\ 1 \end{bmatrix} a$$

Many possible eigenvectors. I'll give two possibilities.

$\boxed{\text{Solution \#1}}$

Multiply $\vec{x}$ by 5

$$\bar{x} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix} a$$

$$\bar{K}_1 = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}$$

$$\bar{K}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$\vec{B}_1 = \mathrm{Re}(\bar{K}_1)$ $\quad$ $\mathrm{Im}(\bar{K}_1) = \vec{B}_2$

$$\vec{X} = C_1 e^{\alpha t}\left[\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t\right] + C_2 e^{\alpha t}\left[\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t\right]$$

$(\alpha = 4, \beta = 1)$

$$\bar{X} = C_1 e^{4t}\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) + C_2 e^{4t}\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right)$$

$\boxed{\text{Solution \#2}}$

Multiply $\vec{x}$ by $2-i$

$$\bar{x} = \begin{bmatrix} \frac{5}{5} \\ 2-i \end{bmatrix} a$$

$$\vec{K}_1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$\bar{K}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i$$

$\vec{B}_1 = \mathrm{Re}(\bar{K}_1)$ $\quad$ $\mathrm{Im}(\bar{K}_1) = \vec{B}_2$

$$\bar{X} = C_1 e^{\alpha t}\left[\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t\right] + C_2 e^{\alpha t}\left[\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t\right]$$

$(\alpha = 4, \beta = 1)$

$$\bar{X} = C_1 e^{4t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) + C_2 e^{4t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t\right)$$

Both solutions are equally acceptable.

(8)

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$-\begin{vmatrix} -1 & 2 \\ 1-\lambda & 0 \end{vmatrix} + (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$-(-2)(1-\lambda) + (1-\lambda)[\underbrace{(1-\lambda)^2-1}_{\lambda^2-2\lambda}] = 0$$

$$(1-\lambda)[2+\lambda^2-2\lambda] = 0$$

$$(1-\lambda)(\lambda^2-2\lambda+2) = 0$$

$$\lambda = 1 \qquad \lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\lambda = \frac{2 \pm \sqrt{-4}}{2}$$

$$\lambda = \frac{2 \pm 2i}{2}$$

$$\lambda = 1 \pm i$$

$\lambda = 1$: $[A - I \mid \vec{0}\,]$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$\frac{R_1}{-1}$ and $\frac{R_2}{-1}$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$

(8) Cont'd

$$R_3 + R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = a$

$x_1 = 0$

$x_2 - 2x_3 = 0 \Rightarrow x_2 = 2a$

$\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} a \qquad \bar{K}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

To deal with the complex roots, find an eigenvector $\bar{K}_2$ for $\lambda = \alpha + \beta i$

$\lambda = 1+i$:

$[A - (1+i)I \mid \bar{0}]$

$$\left[\begin{array}{ccc|c} -i & -1 & 2 & 0 \\ -1 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} -1 & -i & 0 & 0 \\ -i & -1 & 2 & 0 \\ -1 & 0 & -i & 0 \end{array}\right]$$

$$\frac{R_1}{-1} \quad \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ -i & -1 & 2 & 0 \\ -1 & 0 & -i & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 + iR_1 \\ R_3 + R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & i & -i & 0 \end{array}\right]$$

$$\frac{R_2}{-2} \quad \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & i & -i & 0 \end{array}\right] \quad \rightarrow$$

(8) Cont'd

Section 8.2

$$\left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & i & -i & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - iR_2 \\ \\ R_3 - iR_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_3 = a$$

$$x_1 + i x_3 = 0 \Rightarrow x_1 = -ia$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = a$$

$$\bar{x} = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix} a$$

$$\bar{K}_2 = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} i$$

$$\bar{B}_1 = Re(\bar{K}_2) \qquad \vec{B}_2 = Im(\bar{K}_2)$$

$\boxed{\text{Recall } \alpha = 1, \beta = 1}$

$$\vec{X} = C_1 \bar{K}_1 e^{\lambda_1 t} + C_2 e^{\alpha t}(\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t)$$

$$+ C_3 e^{\alpha t}(\bar{B}_1 \sin\beta t + \bar{B}_2 \cos\beta t)$$

$$= C_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^t + C_2 e^t\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \sin t\right)$$

$$+ C_3 e^t\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \cos t\right)$$

OR $\quad \vec{X} = C_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix} e^t + C_3 \begin{bmatrix} -\cos t \\ \sin t \\ \sin t \end{bmatrix} e^t$

OR by scaling last vector:

$$\vec{X} = C_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} \sin t \\ \cos t \\ \cos t \end{bmatrix} e^t + C_3 \begin{bmatrix} \cos t \\ -\sin t \\ -\sin t \end{bmatrix} e^t$$