Section 7.5

① $y' - 3y = \delta(t-2)$, $y(0) = 0$

1) Apply $\mathcal{L}$

$$sY(s) - y(0) - 3Y(s) = e^{-2s}$$

$$(s-3)Y(s) = e^{-2s}$$

2) Solve for $Y(s)$

$$Y(s) = \frac{1}{s-3} e^{-2s}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s-3} e^{-2s} \right\}$$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{s-3} \\ f(t) &= e^{3t} \\ f(t-2) &= e^{3(t-2)} \end{aligned}}$$

$$y(t) = e^{3(t-2)} \mathcal{U}(t-2)$$

(2) $y'' + y = \delta(t - 2\pi)$, $y(0) = 0$, $y'(0) = 1$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = e^{-2\pi s}$$
$$s^2 Y(s) - 1 + Y(s) = e^{-2\pi s}$$

2) Solve for $Y(s)$

$$(s^2+1) Y(s) = 1 + e^{-2\pi s}$$
$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-2\pi s}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \sin t + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} e^{-2\pi s} \right\}$$

$$\boxed{\begin{array}{c} F(s) = \frac{1}{s^2+1} \\ f(t) = \sin t \\ f(t-2\pi) = \sin(t-2\pi) \\ = \sin t \\ \text{because } \sin\theta \\ \text{has period } 2\pi \end{array}}$$

$$y(t) = \sin t + (\sin t)\, u(t-2\pi)$$

Section 7.5

(3) $y'' + y = \delta(t - \frac{\pi}{2}) + \delta(t - \frac{3\pi}{2})$

$y(0) = 0, \quad y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}$$

$$(s^2+1) Y(s) = e^{-\frac{\pi}{2}s} + e^{-\frac{3\pi}{2}s}$$

2) Solve for $Y(s)$

$$Y(s) = \frac{1}{s^2+1} e^{-\frac{\pi}{2}s} + \frac{1}{s^2+1} e^{-\frac{3\pi}{2}s}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} e^{-\frac{\pi}{2}s} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} e^{-\frac{3\pi}{2}s} \right\}$$

$F(s) = \frac{1}{s^2+1}$

$f(t) = \sin t$

$f(t - \frac{\pi}{2}) = \sin t \cos \frac{\pi}{2} - \cos t \sin \frac{\pi}{2}$

$= -\cos t$

$F(s) = \frac{1}{s^2+1}$

$f(t) = \sin t$

$f(t - \frac{3\pi}{2})$

$= \sin t \cos \frac{3\pi}{2} - \cos t \sin \frac{3\pi}{2}$

$= \cos t$

$$y(t) = -\cos t\, \mathcal{U}(t - \tfrac{\pi}{2}) + \cos t\, \mathcal{U}(t - \tfrac{3\pi}{2})$$

(4) $y'' + 2y' = \delta(t-1)$

$y(0) = 0, \quad y'(0) = 1$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + 2[sY(s) - y(0)] = e^{-s}$$

$$s^2 Y(s) - 1 + 2sY(s) = e^{-s}$$

2) Solve for $Y(s)$

$$(s^2 + 2s) Y(s) = 1 + e^{-s}$$

$$Y(s) = \frac{1}{s^2 + 2s}(1 + e^{-s})$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}(1 + e^{-s})\right\}$$

$$\boxed{\begin{aligned} \frac{1}{s(s+2)} &= \frac{A}{s} + \frac{B}{s+2} \\ 1 &= A(s+2) + Bs \\ \text{Sub } s=0:&\quad 1 = 2A \Rightarrow A = \tfrac{1}{2} \\ s=-2:&\quad 1 = -2B \Rightarrow B = \tfrac{-1}{2} \end{aligned}}$$

$$y(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right)e^{-s}\right\}$$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} \\ f(t) &= \frac{1}{2} - \frac{1}{2}e^{-2t} \\ f(t-1) &= \frac{1}{2} - \frac{1}{2}e^{-2(t-1)} \end{aligned}}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left(\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right)u(t-1)$$

(5) $y'' + 4y' + 5y = \delta(t - 2\pi)$

$y(0) = 0, \quad y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4[sY(s) - y(0)] + 5Y(s) = e^{-2\pi s}$$

$$(s^2 + 4s + 5)Y(s) = e^{-2\pi s}$$

2) Solve for $Y(s)$

$$Y(s) = \frac{1}{s^2+4s+5} e^{-2\pi s}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5} e^{-2\pi s}\right\}$$

Complete the Square

$s^2+4s+5 = (s+2)^2 + ?$

$= (s+2)^2 + 1$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1} e^{-2\pi s}\right\}$$

$F(s) = \dfrac{1}{(s+2)^2+1}$

$f(t) = e^{-2t}\sin t$

$f(t-2\pi) = e^{-2(t-2\pi)}\sin(t-2\pi)$

$= e^{-2(t-2\pi)}\sin t$

since $\sin\theta$ has period 2π

$$y(t) = e^{-2(t-2\pi)}\sin t\; u(t-2\pi)$$

(6) $y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$

$y(0) = 1, \ y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + 4[sY(s) - y(0)] + 13Y(s) = e^{-\pi s} + e^{-3\pi s}$$

$$s^2 Y(s) - s + 4s Y(s) - 4 + 13 Y(s) = e^{-\pi s} + e^{-3\pi s}$$

2) Solve for $Y(s)$

$$(s^2 + 4s + 13) Y(s) = s + 4 + e^{-\pi s} + e^{-3\pi s}$$

$$Y(s) = \frac{s+4}{s^2+4s+13} + \frac{1}{s^2+4s+13}\left(e^{-\pi s} + e^{-3\pi s}\right)$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s+4}{s^2+4s+13} + \frac{1}{s^2+4s+13}\left(e^{-\pi s} + e^{-3\pi s}\right)\right\}$$

Complete the Square

$$\begin{aligned} s^2+4s+13 &= (s+2)^2 + ? \\ &= (s+2)^2 + 9 \\ &= (s+2)^2 + 3^2 \end{aligned}$$

$$\begin{aligned} s+4 &= ?(s+2) + ? \\ &= 1(s+2) + 2 \end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s+2}{(s+2)^2+3^2} + \frac{2}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}\left(e^{-\pi s} + e^{-3\pi s}\right)\right\}$$

$\rightarrow$

⑥ Cont'd

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s+2}{(s+2)^2+3^2} + \frac{2}{3}\,\frac{3}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}\left(e^{-\pi s} + e^{-3\pi s}\right) \right\}$$

$$= e^{-2t}\cos 3t + \frac{2}{3}e^{-2t}\sin 3t + \mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2+3^2}\left(e^{-\pi s} + e^{-3\pi s}\right) \right\}$$

$$= e^{-2t}\cos 3t + \frac{2}{3}e^{-2t}\sin 3t + \mathcal{L}^{-1}\left\{ \frac{1}{3}\,\frac{3}{(s+2)^2+3^2}\, e^{-\pi s} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{3}\,\frac{3}{(s+2)^2+3^2}\, e^{-3\pi s} \right\}$$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{3}\,\frac{3}{(s+2)^2+3^2} \\ f(t) &= \frac{1}{3}e^{-2t}\sin 3t \\ f(t-\pi) &= \frac{1}{3}e^{-2(t-\pi)}\sin(3t-3\pi) \\ \text{and } f(t-3\pi) &= \frac{1}{3}e^{-2(t-3\pi)}\sin(3t-9\pi) \end{aligned}}$$

$$= e^{-2t}\cos 3t + \frac{2}{3}e^{-2t}\sin 3t + \frac{1}{3}e^{-2(t-\pi)}\sin(3t-3\pi)\,\mathcal{U}(t-\pi) + \frac{1}{3}e^{-2(t-3\pi)}\sin(3t-9\pi)\,\mathcal{U}(t-3\pi)$$