① $\mathcal{L}\{t\cos 2t\}$

$$= \frac{s^2-4}{(s^2+4)^2}$$

$\boxed{\text{see formula sheet for } \mathcal{L}\{t\cos\omega t\}}$

② $\mathcal{L}\{te^{2t}\sin 6t\}$

$$= \mathcal{L}\{e^{2t}f(t)\}$$

$$= F(s-2)$$

$$= \frac{12(s-2)}{[(s-2)^2+36]^2}$$

$f(t) = t\sin 6t$

$F(s) = \dfrac{12s}{(s^2+36)^2}$

③ $\mathcal{L}\{1 * t^3\}$

$$= F(s)\,G(s)$$

$$= \frac{6}{s^5}$$

$f(t) = 1 \Rightarrow F(s) = \dfrac{1}{s}$

$g(t) = t^3 \Rightarrow G(s) = \dfrac{6}{s^4}$

(4) $\mathcal{L}\{ \int_0^t e^{\tau} d\tau \}$

$= \mathcal{L}\{ \int_0^t e^{\theta} d\theta \}$

$= \mathcal{L}\{ \int_0^t f(\theta) g(t-\theta) \, d\theta$

$= \mathcal{L}\{ f * g \}$

$= F(s)\, G(s)$

$= \dfrac{1}{s(s-1)}$

$f(t) = e^t$
$g(t) = 1$
$F(s) = \frac{1}{s-1}$
$G(s) = \frac{1}{s}$

(5) $\mathcal{L}\{ \int_0^t \tau e^{t-\tau} d\tau \}$

$= \mathcal{L}\{ \int_0^t \theta e^{t-\theta} d\theta \}$

$= \mathcal{L}\{ f * g \}$

$= F(s)\, G(s)$

$= \dfrac{1}{s^2(s-1)}$

$f(t) = t$
$g(t) = e^t$
$F(s) = \frac{1}{s^2}$
$G(s) = \frac{1}{s-1}$

(6) $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$

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$= \mathcal{L}^{-1}\{F(s)G(s)\}$

$$\boxed{\begin{array}{ll} F(s)=\frac{1}{s-1} & G(s)=\frac{1}{s} \\ f(t)=e^t & g(t)=1 \end{array}}$$

$= f * g$

$= \int_0^t f(\theta)g(t-\theta)\,d\theta$

$= \int_0^t e^\theta \,d\theta$

$= e^\theta \Big|_0^t$

$= e^t - 1$

Alternatively:

$$\boxed{\begin{array}{ll} F(s)=\frac{1}{s} & G(s)=\frac{1}{s-1} \\ f(t)=1 & g(t)=e^t \end{array}}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$

$= f * g$

$= \int_0^t f(\theta)g(t-\theta)\,d\theta$

$= \int_0^t e^{t-\theta}\,d\theta$

$= -e^{t-\theta}\Big|_{\theta=0}^{\theta=t}$

$= -1 + e^t$

$= e^t - 1$

(7) $\mathcal{L}^{-1}\left\{ \frac{1}{s^3(s-1)} \right\}$

$= \mathcal{L}^{-1}\{F(s)\,G(s)\}$

$\boxed{\begin{array}{ll} F(s) = \frac{1}{s^3} & G(s) = \frac{1}{s-1} \\ f(t) = \frac{t^2}{2} & g(t) = e^t \end{array}}$

$= f * g$

$= \int_0^t f(\theta)\,g(t-\theta)\,d\theta$

$= \int_0^t \frac{\theta^2}{2} e^{t-\theta}\,d\theta$

	D	I
+	$\theta^2/2$	$e^{t-\theta}$
−	θ	$-e^{t-\theta}$
+	1	$e^{t-\theta}$
		$-e^{t-\theta}$

θ is the variable

$= \left[-\frac{\theta^2}{2} - \theta - 1 \right] e^{t-\theta} \Big|_{\theta=0}^{\theta=t}$

$= -\frac{t^2}{2} - t - 1 + e^t$

$= e^t - \frac{t^2}{2} - t - 1$

Alternatively:

$F(s) = \frac{1}{s-1} \qquad G(s) = \frac{1}{s^3}$

$f(t) = e^t \qquad g(t) = \frac{t^2}{2}$

$\mathcal{L}^{-1}\left\{ \frac{1}{s^3(s-1)} \right\}$

$= f * g$

$= \int_0^t e^{\theta} \frac{(t-\theta)^2}{2}\,d\theta$

$= \int_0^t e^{\theta} \left[\frac{t^2}{2} - \theta t + \frac{\theta^2}{2} \right] d\theta$

$\rightarrow$

⑦ Cont'd

$$= \int_0^t e^{\theta}\left[\frac{t^2}{2} - \theta t + \frac{\theta^2}{2}\right] d\theta$$

	D	I
$+$	$\left(\frac{t^2}{2} - \theta t + \frac{\theta^2}{2}\right)$	e^{θ}
$-$	$(-t + \theta)$	e^{θ}
$+$	1	e^{θ}
		e^{θ}

θ is the variable

$$= \left[\frac{t^2}{2} - \theta t + \frac{\theta^2}{2} + t - \theta + 1\right] e^{\theta} \Big|_{\theta=0}^{\theta=t}$$

$$= \left[\frac{t^2}{2} - t^2 + \frac{t^2}{2} + t - t + 1\right] e^{t} - \left[\frac{t^2}{2} + t + 1\right]$$

$$= e^{t} - \frac{t^2}{2} - t - 1$$

(8) $f(t)$ has period $2a$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\,dt + \int_a^{2a} -e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} e^{-st} \Big|_{t=0}^{t=a} + \frac{1}{s} e^{-st} \Big|_{t=a}^{t=2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{1}{s} e^{-as} + \frac{1}{s} + \frac{1}{s} e^{-2as} - \frac{1}{s} e^{-as} \right]$$

$$= \frac{1}{s} \frac{1}{1-e^{-2as}} \left[1 - 2e^{-as} + e^{-2as} \right]$$

$$= \frac{1}{s} \frac{1}{(1-e^{-as})(1+e^{-as})} (1-e^{-as})^2$$

$$= \frac{1-e^{-as}}{s(1+e^{-as})}$$

A note about the algebra

$$1-x^2 = (1-x)(1+x)$$

$$1-e^{-2as} = 1-(e^{-as})^2 = (1-e^{-as})(1+e^{-as})$$

and

$$1-2m+m^2 = (1-m)^2$$

$$1-2e^{-as} + e^{-2as} = (1-e^{-as})^2$$

(9) $y'' + 9y = \cos 3t$, $y(0) = 2$, $y'(0) = 5$ $\boxed{\text{Section 7.4}}$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + 9Y(s) = \frac{s}{s^2+9}$$

$$s^2 Y(s) - 2s - 5 + 9Y(s) = \frac{s}{s^2+9}$$

$$(s^2+9)Y(s) = 2s+5 + \frac{s}{s^2+9}$$

2) Solve for $Y(s)$

$$Y(s) = \frac{2s+5}{s^2+9} + \frac{s}{(s^2+9)^2}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2s+5}{s^2+9} + \frac{s}{(s^2+9)^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ 2\frac{s}{s^2+9} + \frac{5}{3}\frac{3}{s^2+9} + \frac{1}{6}\frac{6s}{(s^2+9)^2} \right\}$$

$$= 2\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{6}t\sin 3t$$

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(10) $y'' + 16y = f(t)$

$y(0) = 0, \quad y'(0) = 1$

$$\boxed{\begin{array}{l} f(t) = \underset{\text{Initial}}{?} + \underset{\text{change at } t=\pi}{?}\,\mathcal{U}(t-\pi) \\ \\ f(t) = \cos 4t - \cos 4t\,\mathcal{U}(t-\pi) \end{array}}$$

$y'' + 16y = \cos 4t - \cos 4t\,\mathcal{U}(t-\pi)$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{s}{s^2+16} - \mathcal{L}\{\cos 4t\,\mathcal{U}(t-\pi)\}$$

$$s^2Y(s) - 1 + 16Y(s) = \frac{s}{s^2+16} - \frac{s}{s^2+16}e^{-\pi s}$$

$$\boxed{\begin{array}{l} g(t) = \cos 4t \\ g(t+\pi) = \cos(4t+4\pi) \\ \quad = \cos 4t \\ \text{since } \cos\theta \\ \text{has period } 2\pi \\ \mathcal{L}\{g(\cancel{t}t+\pi)\} = \dfrac{s}{s^2+16} \end{array}}$$

2) Solve for $Y(s)$

$$(s^2+16)Y(s) = 1 + \frac{s}{s^2+16} - \frac{s}{s^2+16}e^{-\pi s}$$

$$Y(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}$$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}\right\}$$

$\rightarrow$

⑩ Cont'd Section 7.4

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+16} + \frac{s}{(s^2+16)^2} - \frac{s}{(s^2+16)^2}e^{-\pi s}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{4}{s^2+16} + \frac{1}{8}\frac{8s}{(s^2+16)^2} - \frac{1}{8}\frac{8s}{(s^2+16)^2}e^{-\pi s}\right\}$$

$$= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}h(t-\pi)\mathcal{U}(t-\pi)$$

$$\boxed{\begin{aligned} H(s) &= \frac{8s}{(s^2+16)^2} \\ h(t) &= t\sin 4t \end{aligned}}$$

$$= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t - \frac{1}{8}(t-\pi)\sin(4t-4\pi)\,\mathcal{U}(t-\pi)$$