Section 7.3

① $\mathcal{L}\{te^{10t}\}$

$= F(s-10)$

$$\boxed{\begin{array}{l} f(t)=t \\ F(s)=\dfrac{1}{s^2} \end{array}}$$

$= \dfrac{1}{(s-10)^2}$

② $\mathcal{L}\{t^3e^{-2t}\}$

$= F(s+2)$

$$\boxed{\begin{array}{l} f(t)=t^3 \\ F(s)=\dfrac{6}{s^4} \end{array}}$$

$= \dfrac{6}{(s+2)^4}$

③ $\mathcal{L}\{e^t \sin 3t\}$

$= F(s-1)$

$$\boxed{\begin{array}{l} f(t)=\sin 3t \\ F(s)=\dfrac{3}{s^2+9} \end{array}}$$

$= \dfrac{3}{(s-1)^2+9}$

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(4) $\mathcal{L}\{(1-e^{t}+3e^{-4t})\cos 5t\}$

$= F(s) - F(s-1) + 3F(s+4)$

$f(t) = \cos 5t$
$F(s) = \dfrac{s}{s^2+25}$

$= \dfrac{s}{s^2+25} - \dfrac{(s-1)}{(s-1)^2+25} + \dfrac{3(s+4)}{(s+4)^2+25}$

(5) $\mathcal{L}^{-1}\{\dfrac{1}{(s+2)^3}\}$

$= e^{-2t} f(t)$

$F(s) = \dfrac{1}{s^3}$
$= \dfrac{1}{2}\dfrac{2!}{s^3}$
$f(t) = \dfrac{t^2}{2}$

$= \frac{1}{2} t^2 e^{-2t}$

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(6) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$

$$\boxed{\begin{aligned}&\text{Complete the Square}\\ &s^2-6s+10 = (s-3)^2+\ ?\\ &\qquad\qquad\quad = (s-3)^2+1\end{aligned}}$$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$

$= e^{3t} f(t)$

$$\boxed{\begin{aligned}&F(s) = \frac{1}{s^2+1}\\ &f(t) = \sin t\end{aligned}}$$

$= e^{3t}\sin t$

(7) $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\}$

$= e^{-t} f(t)$

$$\boxed{\begin{aligned}&F(s) = \frac{1}{s} - \frac{1}{s^2}\\ &f(t) = 1 - t\end{aligned}}$$

$= (1-t)e^{-t}$

or $e^{-t} - te^{-t}$

(8) $y'' - 6y' + 9y = t$ Section 7.3

$y(0) = 0$, $y'(0) = 1$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - 1 - 6sY(s) + 9Y(s) = \frac{1}{s^2}$$

2) Solve for $Y(s)$:

$$(s^2 - 6s + 9)Y(s) = 1 + \frac{1}{s^2}$$

$$(s-3)^2 Y(s) = 1 + \frac{1}{s^2}$$

$$Y(s) = \frac{1}{(s-3)^2} + \frac{1}{s^2(s-3)^2}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2} + \frac{1}{s^2(s-3)^2}\right\}$$

Partial Fractions

$$\frac{1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$1 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2$$

Sub $s=0$: $1 = 9B \Rightarrow B = \frac{1}{9}$

$s=3$: $1 = 9D \Rightarrow D = \frac{1}{9}$

s^3 coefficient: $0 = A + C$ (1)

s coefficient: $0 = 9A - 6B$

$0 = 9A - \frac{6}{9} \Rightarrow A = \frac{6}{81} = \frac{2}{27}$

$A = \frac{2}{27} \to$ (1): $C = \frac{-2}{27}$

$\rightarrow$

8 Cont'd

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2} + \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{1}{9}\frac{1}{(s-3)^2}\right\}$$

$$y(t) = \frac{2}{27} + \frac{1}{9}t + \mathcal{L}^{-1}\left\{\frac{10}{9}\frac{1}{(s-3)^2} - \frac{2}{27}\frac{1}{s-3}\right\}$$

$$y(t) = \frac{2}{27} + \frac{1}{9}t + e^{3t}f(t)$$

$$\boxed{\begin{array}{l} F(s) = \frac{10}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s} \\ f(t) = \frac{10}{9}t - \frac{2}{27} \end{array}}$$

$$y(t) = \frac{2}{27} + \frac{1}{9}t + e^{3t}\left(\frac{10}{9}t - \frac{2}{27}\right)$$

or $y(t) = \frac{1}{9}t + \frac{2}{27} - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}$

(9) $y'' - 6y' + 13y = 0$

$y(0) = 0, \; y'(0) = -3$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) - 6[sY(s) - y(0)] + 13\,Y(s) = 0$$

$$s^2 Y(s) \quad + 3 \quad - 6sY(s) \quad + 13\,Y(s) = 0$$

2) Solve for $Y(s)$:

$$(s^2 - 6s + 13)Y(s) = -3$$

$$Y(s) = \frac{-3}{s^2 - 6s + 13}$$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-3}{s^2 - 6s + 13} \right\}$$

Complete the Square

$$s^2 - 6s + 13 = (s-3)^2 + ?$$

$$= (s-3)^2 + 4$$

$$= (s-3)^2 + 2^2$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-3}{(s-3)^2 + 2^2} \right\}$$

$$y(t) = e^{3t} f(t)$$

$$F(s) = \frac{-3}{s^2 + 2^2}$$

$$= \frac{-3}{2}\left(\frac{2}{s^2 + 2^2}\right)$$

$$f(t) = \frac{-3}{2} \sin 2t$$

$$y(t) = \frac{-3}{2} e^{3t} \sin 2t$$

(10) $y'' - y' = e^t \cos t$ Section 7.3

$y(0) = 0, \quad y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) - [sY(s) - y(0)] = \mathcal{L}\{e^t \cos t\}$$

$$(s^2 - s) Y(s) = \mathcal{L}\{e^t \cos t\}$$

$$(s^2 - s) Y(s) = F(s-1)$$

$$\boxed{\begin{aligned} f(t) &= \cos t \\ F(s) &= \frac{s}{s^2+1} \end{aligned}}$$

$$(s^2 - s) Y(s) = \frac{s-1}{(s-1)^2+1}$$

2) Solve for $Y(s)$:

$$Y(s) = \left[\frac{s-1}{(s-1)^2+1}\right] \frac{1}{s^2 - s}$$

Prepare for partial fractions

$$Y(s) = \frac{s-1}{(s^2 - 2s + 2)\, s(s-1)}$$

$$Y(s) = \frac{1}{(s^2 - 2s + 2)\, s}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2 - 2s + 2)\, s}\right\} \rightarrow$$

(10) Cont'd

Partial Fractions $\dfrac{1}{(s^2-2s+2)s} = \dfrac{As+B}{s^2-2s+2} + \dfrac{C}{s}$

$1 = (As+B)s + C(s^2-2s+2)$

Sub $s=0$: $1 = 2C \Rightarrow C = \frac{1}{2}$

s^2 coefficient: $0 = A + C \Rightarrow A = -\frac{1}{2}$

s coefficient: $0 = B - 2C$

$0 = B - 1$

$B = 1$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{2}s+1}{s^2-2s+2} + \frac{1}{2}\cdot\frac{1}{s} \right\}$$

$$y(t) = \frac{1}{2} + \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{2}s+1}{s^2-2s+2} \right\}$$

Complete the Square

$$= \frac{1}{2} + \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{2}s+1}{(s-1)^2+1} \right\}$$

$-\frac{1}{2}s+1 = ?(s-1) + ?$

$= -\frac{1}{2}(s-1) + ?$

$= -\frac{1}{2}(s-1) + \frac{1}{2}$

$$= \frac{1}{2} + \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{2}(s-1)+\frac{1}{2}}{(s-1)^2+1} \right\}$$

$$= \frac{1}{2} + e^t f(t)$$

$F(s) = \dfrac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$

$f(t) = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$

$$= \frac{1}{2} - \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t$$

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(11) $\mathcal{L}\{t\,\mathcal{U}(t-2)\}$

$= e^{-2s}\,\mathcal{L}\{f(t+2)\}$

$$\boxed{\begin{aligned} f(t) &= t \\ f(t+2) &= t+2 \end{aligned}}$$

$= e^{-2s}\,\mathcal{L}\{t+2\}$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$

or $\quad \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$

(12) $\mathcal{L}\{(\cos 2t)\,\mathcal{U}(t-\pi)\}$

$= e^{-\pi s}\,\mathcal{L}\{f(t+\pi)\}$

$$\boxed{\begin{aligned} f(t) &= \cos 2t \\ f(t+\pi) &= \cos 2(t+\pi) \\ &= \cos(2t+2\pi) \\ &= \cos 2t \\ &\text{Because the function } \cos\theta \\ &\text{has period } 2\pi \end{aligned}}$$

$= e^{-\pi s}\,\mathcal{L}\{\cos 2t\}$

$= e^{-\pi s}\,\dfrac{s}{s^2+4}$

or $\quad \dfrac{s}{s^2+4}\,e^{-\pi s}$

(13) $\mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s^3} \right\}$

$= f(t-2)\,\mathcal{U}(t-2)$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{s^3} \\ &= \frac{1}{2}\,\frac{2!}{s^3} \\ f(t) &= \tfrac{1}{2}t^2 \end{aligned}}$$

$= \frac{1}{2}(t-2)^2\,\mathcal{U}(t-2)$

(14) $\mathcal{L}^{-1}\left\{ \frac{e^{-\pi s}}{s^2+1} \right\}$

$= f(t-\pi)\,\mathcal{U}(t-\pi)$

$F(s) = \frac{1}{s^2+1}$

$f(t) = \sin t$

$f(t-\pi) = \sin(t-\pi)$

OR

Use $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$f(t-\pi) = \sin(t-\pi)$

$= \sin t\cos\pi - \cos t\sin\pi$

$= -\sin t$

$= \sin(t-\pi)\,\mathcal{U}(t-\pi)$

or $-\sin t\,\mathcal{U}(t-\pi)$

(15) $\mathcal{L}^{-1}\left\{ \dfrac{e^{-s}}{s(s+1)} \right\}$

$= \mathcal{L}^{-1}\left\{ \dfrac{1}{s(s+1)}\, e^{-s} \right\}$

$$\boxed{\begin{aligned} \frac{1}{s(s+1)} &= \frac{A}{s} + \frac{B}{s+1} \\ 1 &= A(s+1) + Bs \\ \text{Sub } s=0 &: \quad 1 = A \\ s=-1 &: \quad 1 = -B \Rightarrow B = -1 \\ \frac{1}{s(s+1)} &= \frac{1}{s} - \frac{1}{s+1} \end{aligned}}$$

$= \mathcal{L}^{-1}\left\{ \left(\frac{1}{s} - \frac{1}{s+1} \right) e^{-s} \right\}$

$= f(t-1)\,\mathcal{U}(t-1)$

$$\boxed{\begin{aligned} F(s) &= \frac{1}{s} - \frac{1}{s+1} \\ f(t) &= 1 - e^{-t} \end{aligned}}$$

$= \left(1 - e^{-(t-1)}\right) \mathcal{U}(t-1)$

or $\mathcal{U}(t-1) - e^{-(t-1)}\,\mathcal{U}(t-1)$

(16) $f(t) = \underset{\text{Initial}}{?} + \underset{\text{Change at } t=3}{?}\,\mathcal{U}(t-3)$

$f(t) = 2 - 4\,\mathcal{U}(t-3)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{2 - 4\,\mathcal{U}(t-3)\}$

$= \frac{2}{s} - 4\,\mathcal{L}\{1\,\mathcal{U}(t-3)\}$

$= \frac{2}{s} - \frac{4}{s}e^{-3s}$

$$\boxed{\begin{aligned} g(t) &= 1 \\ g(t+3) &= 1 \\ \mathcal{L}\{g(t+3)\} &= \tfrac{1}{s} \end{aligned}}$$

(17) $f(t) = \underset{\text{Initial}}{?} + \underset{\text{Change at } t=1}{?}\,\mathcal{U}(t-1)$

$f(t) = 0 + t^2\,\mathcal{U}(t-1)$

$= t^2\,\mathcal{U}(t-1)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\,\mathcal{U}(t-1)\}$

$= \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)e^{-s}$

$$\boxed{\begin{aligned} g(t) &= t^2 \\ g(t+1) &= (t+1)^2 \\ &= t^2+2t+1 \\ \mathcal{L}\{g(t+1)\} &= \tfrac{2}{s^3} + \tfrac{2}{s^2} + \tfrac{1}{s} \end{aligned}}$$

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(18) $f(t) = \underset{\substack{\uparrow \\ \text{Initial}}}{?} + \underset{\substack{\uparrow \\ \text{Change at} \\ t=2}}{?}\, \mathcal{U}(t-2)$

$f(t) = t - t\,\mathcal{U}(t-2)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t - t\,\mathcal{U}(t-2)\}$

$= \frac{1}{s^2} - \mathcal{L}\{t\,\mathcal{U}(t-2)\}$

$$\boxed{\begin{array}{c} g(t) = t \\ g(t+2) = t+2 \\ \mathcal{L}\{g(t+2)\} = \frac{1}{s^2} + \frac{2}{s} \end{array}}$$

$= \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{2}{s}\right) e^{-2s}$

or $\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$

(19) $y' + y = f(t)$, $y(0) = 0$

$f(t) = \underset{\text{Initial}}{?} + \underset{\substack{\text{Change} \\ \text{at } t=1}}{?}\, \mathcal{U}(t-1)$

$f(t) = 0 + 5\,\mathcal{U}(t-1)$
$= 5\,\mathcal{U}(t-1)$

$y' + y = 5\,\mathcal{U}(t-1)$, $y(0) = 0$

1) Apply $\mathcal{L}$

$sY(s) - y(0) + Y(s) = \mathcal{L}\{5\,\mathcal{U}(t-1)\}$

$g(t) = 5$
$g(t+1) = 5$
$\mathcal{L}\{g(t+1)\} = \frac{5}{s}$

$sY(s) + Y(s) = \frac{5}{s} e^{-s}$

2) Solve for $Y(s)$

$(s+1)\,Y(s) = \frac{5}{s} e^{-s}$

$Y(s) = \frac{5}{s(s+1)} e^{-s}$

3) Apply $\mathcal{L}^{-1}$

$y(t) = \mathcal{L}^{-1}\left\{ \frac{5}{s(s+1)} e^{-s} \right\}$

$\rightarrow$

(19) Cont'd

$$\boxed{\begin{aligned} &\text{Partial Fractions} \quad \frac{5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \\ &\qquad 5 = A(s+1) + Bs \\ &\text{Sub } s=0: \ 5 = A \\ &\qquad s=-1: \ 5 = -B \Rightarrow B = -5 \end{aligned}}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \left(\frac{5}{s} - \frac{5}{s+1} \right) e^{-s} \right\}$$

$$= h(t-1)\,\mathcal{U}(t-1)$$

$$\boxed{\begin{aligned} H(s) &= \frac{5}{s} - \frac{5}{s+1} \\ h(t) &= 5 - 5e^{-t} \end{aligned}}$$

$$= \left(5 - 5e^{-(t-1)}\right)\mathcal{U}(t-1)$$

(20) $y'' + 4y = (\sin t)\,\mathcal{U}(t-2\pi)$ Section 7.3

$y(0) = 1, \quad y'(0) = 0$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = \mathcal{L}\{(\sin t)\,\mathcal{U}(t-2\pi)\}$$

$g(t) = \sin t$

$g(t+2\pi) = \sin(t+2\pi)$

$= \sin t$

because period of $\sin\theta$ is 2π

$\mathcal{L}\{g(t+2\pi)\} = \frac{1}{s^2+1}$

$$s^2 Y(s) - s + 4Y(s) = e^{-2\pi s}\frac{1}{s^2+1}$$

2) Solve for $Y(s)$

$$(s^2+4)Y(s) = s + \frac{e^{-2\pi s}}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+4} + \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\}$$

$\rightarrow$

20 Cont'd

$$y(t) = \cos 2t + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}\right\}$$

Partial Fractions

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

Sub $s = i$: $1 = (Ai+B)(3)$

$1 = 3Ai + 3B$

$A = 0$

$B = \frac{1}{3}$

Sub $s = 2i$: $1 = (2iC + D)(-3)$

$1 = -6Ci - 3D$

$C = 0$

$D = \frac{-1}{3}$

$$y(t) = \cos 2t + \mathcal{L}^{-1}\left\{\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}\right)e^{-2\pi s}\right\}$$

$$= \cos 2t + \mathcal{L}^{-1}\left\{\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)e^{-2\pi s}\right\}$$

$F(s) = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}$

$f(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

$f(t-2\pi) = \frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)$

$$= \cos 2t + \left[-\frac{1}{6}\sin(2t-4\pi) + \frac{1}{3}\sin(t-2\pi)\right]\mathcal{U}(t-2\pi)$$

OR $y(t) = \cos 2t + \left[-\frac{1}{6}\sin 2t + \frac{1}{3}\sin t\right]\mathcal{U}(t-2\pi)$

because $\sin\theta$ has period 2π.