Section 6.2

(9) $y'' - 2xy' + y = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\boxed{\begin{array}{c} k = n-2 \\ n = k+2 \\ n=2 \Rightarrow k=0 \end{array}} \qquad \boxed{k=n} \qquad \boxed{k=n}$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} 2kc_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$\boxed{\text{Start at largest k-value : } k=1}$

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} 2kc_k x^k + c_0 + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$(2c_2 + c_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - 2kc_k + c_k\right]x^k = 0$$

$$2c_2 + c_0 = 0 \Rightarrow c_2 = -\frac{c_0}{2}$$

$$(k+2)(k+1)c_{k+2} - 2kc_k + c_k = 0$$

$$\Rightarrow c_{k+2} = \frac{(2k-1)c_k}{(k+2)(k+1)} \quad \text{for } k \geq 1$$

(9) Cnt'd

Recopying:
$$C_{k+2} = \frac{(2k-1)C_k}{(k+2)(k+1)} \quad \text{for } k \geq 1$$

$$(k=1) \qquad C_3 = \frac{C_1}{3\cdot 2} = \frac{C_1}{6}$$

$$(k=2) \qquad C_4 = \frac{3C_2}{4\cdot 3} = \frac{C_2}{4} = -\frac{C_0}{8}$$

$$(k=3) \qquad C_5 = \frac{5C_3}{5\cdot 4} = \frac{C_3}{4} = \frac{1}{4}\left(\frac{C_1}{6}\right) = \frac{C_1}{24}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots$$

$$= C_0 + C_1 x - \frac{C_0}{2}x^2 + \frac{C_1}{6}x^3 - \frac{C_0}{8}x^4 + \frac{C_1}{24}x^5 \ldots$$

$$= C_0\Big[\overbrace{1 - \frac{x^2}{2} - \frac{x^4}{8} + \ldots}^{y_1}\Big]$$

$$+ C_1\Big[\underbrace{x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots}_{y_2}\Big]$$

(11) $y'' + x^2 y' + xy = 0$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n C_n x^{n-1} + x \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^{n+1} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\boxed{\begin{matrix} k=n-2 \\ n=k+2 \\ n=2 \Rightarrow k=0 \end{matrix}} \qquad \boxed{\begin{matrix} k=n+1 \\ n=k-1 \\ n=1 \Rightarrow k=2 \end{matrix}} \qquad \boxed{\begin{matrix} k=n+1 \\ n=k-1 \\ n=0 \Rightarrow k=1 \end{matrix}}$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2} x^k + \sum_{k=2}^{\infty} (k-1)C_{k-1} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$\boxed{\text{Start at largest } k\text{-value} : k=2}$

$$2C_2 + 6C_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) C_{k+2} x^k$$

$$+ \sum_{k=2}^{\infty} (k-1)C_{k-1} x^k + C_0 x + \sum_{k=2}^{\infty} C_{k-1} x^k = 0$$

$$\underbrace{2C_2 + 6C_3 x + C_0 x}_{2C_2 + (6C_3 + C_0)x} + \sum_{k=2}^{\infty} \left[(k+2)(k+1)C_{k+2} + (k-1)C_{k-1} + C_{k-1}\right] x^k = 0$$

$$2C_2 = 0 \Rightarrow C_2 = 0$$

$$6C_3 + C_0 = 0 \Rightarrow C_3 = \frac{-C_0}{6}$$

$$(k+2)(k+1)C_{k+2} + (k-1)C_{k-1} + C_{k-1} = 0 \Rightarrow C_{k+2} = \frac{-k C_{k-1}}{(k+2)(k+1)}$$

for $k \geq 2$ $\longrightarrow$

(11) Cont'd

$$C_{k+2} = \frac{-k\,C_{k-1}}{(k+2)(k+1)} \quad \text{for } k \geq 2$$

$(k=2)$ $\quad C_4 = \frac{-2C_1}{4\cdot 3} = \frac{-C_1}{6}$

$(k=3)$ $\quad C_5 = \frac{-3C_2}{5\cdot 4} = \frac{-3}{20}(0) = 0$

$(k=4)$ $\quad C_6 = \frac{-4C_3}{6\cdot 5} = \frac{-2}{15}\left(\frac{-C_0}{6}\right) = \frac{C_0}{45}$

$(k=5)$ $\quad C_7 = \frac{-5C_4}{7\cdot 6} = \frac{-5}{42}\left(\frac{-C_1}{6}\right) = \frac{5C_1}{252}$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$= C_0 + C_1 x - \frac{C_0}{6}x^3 - \frac{C_1}{6}x^4 + \frac{C_0}{45}x^6 + \frac{5C_1}{252}x^7 + \dots$$

$$= C_0\Big[\overbrace{1 - \frac{x^3}{6} + \frac{x^6}{45} + \dots}^{y_1}\Big]$$

$$+ C_1\Big[\underbrace{x - \frac{x^4}{6} + \frac{5}{252}x^7 + \dots}_{y_2}\Big]$$

(13) $(x-1)y'' + y' = 0$

$$(x-1)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=1}^{\infty} nC_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=1}^{\infty} nC_n x^{n-1} = 0$$

$\boxed{\begin{array}{c} k=n-1 \\ n=k+1 \\ n=2 \Rightarrow k=1 \end{array}}$ $\boxed{\begin{array}{c} k=n-2 \\ n=k+2 \\ n=2 \Rightarrow k=0 \end{array}}$ $\boxed{\begin{array}{c} k=n-1 \\ n=k+1 \\ n=1 \Rightarrow k=0 \end{array}}$

$$\sum_{k=1}^{\infty}(k+1)kC_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)C_{k+1}x^k = 0$$

$\boxed{\text{Start at largest k-value : } k=1}$

$$\sum_{k=1}^{\infty}(k+1)kC_{k+1}x^k - 2C_2 - \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k + C_1$$

$$+\sum_{k=1}^{\infty}(k+1)C_{k+1}x^k = 0$$

$$C_1 - 2C_2 + \sum_{k=1}^{\infty}\left[(k+1)kC_{k+1} - (k+2)(k+1)C_{k+2} + (k+1)C_{k+1}\right]x^k = 0$$

$$C_1 - 2C_2 = 0 \Rightarrow C_2 = \frac{C_1}{2}$$

$$(k+1)kC_{k+1} - (k+2)(k+1)C_{k+2} + (k+1)C_{k+1} = 0$$

$$-(k+2)(k+1)C_{k+2} = -(k+1)kC_{k+1} - (k+1)C_{k+1}$$

$$(k+2)C_{k+2} = kC_{k+1} + C_{k+1}$$

$$C_{k+2} = \frac{(k+1)C_{k+1}}{(k+2)} \quad \text{for } k \geq 1$$

(13) Cont'd $\quad C_{k+2} = \dfrac{(k+1)C_{k+1}}{(k+2)}$ for $k \geq 1$

$(k=1) \quad C_3 = \dfrac{2C_2}{3} = \dfrac{2}{3}\left(\dfrac{C_1}{2}\right) = \dfrac{C_1}{3}$

$(k=2) \quad C_4 = \dfrac{3C_3}{4} = \dfrac{3}{4}\left(\dfrac{C_1}{3}\right) = \dfrac{C_1}{4}$

$(k=3) \quad C_5 = \dfrac{4C_4}{5} = \dfrac{4}{5}\left(\dfrac{C_1}{4}\right) = \dfrac{C_1}{5}$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$= C_0 + C_1 x + \frac{C_1}{2}x^2 + \frac{C_1}{3}x^3 + \frac{C_1}{4}x^4 + \frac{C_1}{5}x^5 + \dots$$

y_1

$$= C_0 [1]$$

$$+ C_1 \Big[\underbrace{x + \frac{x^2}{2} + \frac{x^3}{3} + \dots}_{y_2}\Big]$$

(15) $y'' - (x+1)y' - y = 0$

$$y'' - xy' - y' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$- \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$\boxed{\begin{matrix} k=n-2 \\ n=k+2 \\ n=2 \Rightarrow k=0 \end{matrix}}$ $\boxed{k=n}$ $\boxed{\begin{matrix} k=n-1 \\ n=k+1 \\ n=1 \Rightarrow k=0 \end{matrix}}$ $\boxed{k=n}$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$\boxed{\text{Start at largest k-value : } k=1}$

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k$$

$$- c_1 - \sum_{k=1}^{\infty} (k+1)c_{k+1} x^k - c_0 - \sum_{k=1}^{\infty} c_k x^k = 0$$

$$2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - k c_k - (k+1)c_{k+1} - c_k\right] x^k = 0$$

(15) Cont'd

$$2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - (k+1)c_k - (k+1)c_{k+1}\right] x^k = 0$$

$$2c_2 - c_1 - c_0 = 0 \Rightarrow c_2 = \frac{c_1 + c_0}{2} = \frac{c_0}{2} + \frac{c_1}{2}$$

$$(k+2)(k+1)c_{k+2} - (k+1)c_k - (k+1)c_{k+1} = 0$$

$$\Rightarrow c_{k+2} = \frac{(k+1)c_k + (k+1)c_{k+1}}{(k+2)(k+1)}$$

$$= \frac{c_k + c_{k+1}}{k+2} \quad \text{for } k \geq 1$$

$$(k=1) \quad c_3 = \frac{c_1 + c_2}{3}$$

$$= \frac{c_1}{3} + \frac{c_2}{3}$$

$$= \frac{c_1}{3} + \frac{1}{3}\left(\frac{c_0}{2} + \frac{c_1}{2}\right)$$

$$= \frac{c_1}{3} + \frac{c_0}{6} + \frac{c_1}{6}$$

$$= \frac{c_1}{2} + \frac{c_0}{6}$$

$$y = c_0 + c_1 x + c_2 x^2 + \dots$$

$$= c_0 + c_1 x + \left(\frac{c_0}{2} + \frac{c_1}{2}\right)x^2 + \left(\frac{c_0}{6} + \frac{c_1}{2}\right)x^3 + \dots$$

$$= c_0 \underbrace{\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right]}_{y_1} + c_1 \underbrace{\left[x + \frac{x^2}{2} + \frac{x^3}{2} + \dots\right]}_{y_2}$$

19 $(x-1)y'' - xy' + y = 0 \qquad y(0) = -2$

$y'(0) = 6$

$$xy'' - y'' - xy' + y = 0$$

$$x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$-x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$\boxed{\begin{array}{c} k=n-1 \\ n=k+1 \\ n=2 \Rightarrow k=1 \end{array}}$ $\boxed{\begin{array}{c} k=n-2 \\ n=k+2 \\ n=2 \Rightarrow k=0 \end{array}}$ $\boxed{k=n}$ $\boxed{k=n}$

$$\sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_k x^k + \sum_{k=0}^{\infty}c_k x^k = 0$$

$\boxed{\text{Start at largest } k\text{-value}: k=1}$

$$\sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - 2c_2 - \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k$$

$$-\sum_{k=1}^{\infty}kc_k x^k + c_0 + \sum_{k=1}^{\infty}c_k x^k = 0$$

$$-2c_2 + c_0 + \sum_{k=1}^{\infty}\left[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} - kc_k + c_k\right]x^k = 0$$

(19) Cont'd

$$-2C_2 + C_0 = 0 \Rightarrow C_2 = \frac{C_0}{2}$$

$$(k+1)k\,C_{k+1} - (k+2)(k+1)C_{k+2} - kC_k + C_k = 0$$

$$\Rightarrow -(k+2)(k+1)C_{k+2} = (k-1)C_k - (k+1)k\,C_{k+1}$$

$$C_{k+2} = \frac{(k+1)k\,C_{k+1} - (k-1)C_k}{(k+2)(k+1)}$$

for $k \geq 1$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y(0) = -2 \Rightarrow \boxed{-2 = C_0}$$

$$y' = C_1 + 2C_2 x + \cdots$$

$$y'(0) = 6 \Rightarrow \boxed{6 = C_1}$$

$$C_2 = \frac{C_0}{2} = -1$$

$$(k=1) \quad C_3 = \frac{2\cdot 1\,C_2 - 0C_1}{3\cdot 2} = \frac{-1}{3}$$

$$(k=2) \quad C_4 = \frac{3\cdot 2C_3 - 1C_2}{4\cdot 3} = \frac{-2+1}{12} = \frac{-1}{12}$$

$$(k=3) \quad C_5 = \frac{4\cdot 3C_4 - 2C_3}{5\cdot 4} = \frac{-1+\frac{2}{3}}{20} = \frac{-3+2}{60} = \frac{-1}{60}$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$= -2 + 6x - x^2 - \frac{x^3}{3} - \frac{x^4}{12} - \frac{x^5}{60} + \cdots$$

(21) $y'' - 2xy' + 8y = 0$ $\quad y(0) = 3$

$y'(0) = 0$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 2x \sum_{n=1}^{\infty} n C_n x^{n-1} + 8 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} 2n C_n x^n + \sum_{n=0}^{\infty} 8 C_n x^n = 0$$

$$\boxed{\begin{array}{c} k = n-2 \\ n = k+2 \\ n = 2 \Rightarrow k = 0 \end{array}} \qquad \boxed{k = n} \qquad \boxed{k = n}$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2} x^k - \sum_{k=1}^{\infty} 2k C_k x^k + \sum_{k=0}^{\infty} 8 C_k x^k = 0$$

| Start at largest k-value : k=1 |

$$2C_2 + \sum_{k=1}^{\infty} (k+2)(k+1)C_{k+2} x^k - \sum_{k=1}^{\infty} 2k C_k x^k$$

$$+ 8C_0 + \sum_{k=1}^{\infty} 8 C_k x^k = 0$$

$$2C_2 + 8C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - 2k C_k + 8 C_k \right] x^k = 0$$

(21) Cont'd

$$2C_2 + 8C_0 = 0 \Rightarrow C_2 = -4C_0$$

$$(k+2)(k+1)C_{k+2} - 2kC_k + 8C_k = 0$$

$$\Rightarrow C_{k+2} = \frac{2kC_k - 8C_k}{(k+2)(k+1)}$$

for $k \geq 1$

$(k=1)$

$$C_3 = \frac{2C_1 - 8C_1}{3 \cdot 2} = -C_1$$

$(k=2)$

$$C_4 = \frac{4C_2 - 8C_2}{4 \cdot 3} = \frac{-C_2}{3} = \frac{4C_0}{3}$$

$(k=3)$

$$C_5 = \frac{6C_3 - 8C_3}{5 \cdot 4} = \frac{-C_3}{10} = \frac{C_1}{10}$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$3 = y$
$x = 0$ $\Rightarrow$ $3 = C_0$ $\rightarrow$

(21) Cont'd

$$y' = C_1 + 2C_2x + \cdots$$

$\begin{matrix} y' = 0 \\ x = 0 \end{matrix} \Rightarrow 0 = C_1$

$$C_2 = -4C_0 = -12$$

$$C_3 = -C_1 = 0$$

$$C_4 = \frac{4}{3}C_0 = 4$$

$$C_5 = \frac{C_1}{10} = 0$$

$$y = C_0 + C_1x + C_2x^2 + \cdots$$
$$= 3 \quad -12x^2 + 4x^4 + \cdots$$

(23) $y'' + (\sin x)y = 0$

$$\begin{cases} y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \dots \\ y' = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^4 + 6C_6 x^5 + \dots \\ y'' = 2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + 30C_6 x^4 + \dots \\ \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \end{cases}$$

Let's include all terms up to, and including, x^4 (arbitrary choice)

$$2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + 30C_6 x^4 + \dots$$
$$+ \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots\right) = 0$$

$$\left.\begin{matrix} 2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + 30C_6 x^4 + \dots \\ C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + \dots \\ - \frac{C_0 x^3}{6} - \frac{C_1 x^4}{6} + \dots \end{matrix}\right\} = 0$$

All coefficients are $= 0$

$$2C_2 = 0 \Rightarrow C_2 = 0$$
$$6C_3 + C_0 = 0 \Rightarrow C_3 = \frac{-C_0}{6}$$
$$12C_4 + C_1 = 0 \Rightarrow C_4 = -\frac{C_1}{12} \quad \rightarrow$$

$$20C_5 + C_2 - \frac{C_0}{6} = 0$$

$$\Rightarrow C_5 = -\frac{C_2}{20} + \frac{C_0}{120}$$

$$= \frac{C_0}{120}$$

$$30C_6 + C_3 - \frac{C_1}{6} = 0 \Rightarrow C_6 = \frac{-C_3}{30} + \frac{C_1}{180}$$

$$= \frac{C_0}{180} + \frac{C_1}{180}$$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$= C_0 + C_1 x - \frac{C_0}{6} x^3 - \frac{C_1}{12} x^4 + \frac{C_0}{120} x^5 + \left(\frac{C_0}{180} + \frac{C_1}{180}\right) x^6 + \dots$$

$$= C_0 \Big[\overbrace{1 - \frac{x^3}{6} + \frac{x^5}{120} + \dots}^{y_1} \Big]$$

$$+ C_1 \Big[\underbrace{x - \frac{x^4}{12} + \frac{x^6}{180} + \dots}_{y_2} \Big]$$