(11) a) $\boxed{\text{Section 5.1}}$

$$m x'' + \beta x' + k x = f(t)$$

$\boxed{m = \dfrac{19.6 \text{ N}}{9.8 \text{ N/kg}} = 2 \text{ kg}}$ $\quad \beta = 0 \quad$ $\boxed{\begin{array}{c} 19.6 \text{ N} = k\,(0.098 \text{ m}) \\ k = 200 \text{ N/m} \end{array}}$ $\quad f(t) = 0$

$$2x'' + 200x = 0$$

or $\quad x'' + 100x = 0$

Initial Conditions

$x(0) = -\frac{2}{3}$ m $\quad$ ↑ ⊖

$x'(0) = 5$ m/s $\quad$ ↓ ⊕

$$m^2 + 100 = 0$$
$$m^2 = -100$$
$$m = \pm\sqrt{-100}$$
$$m = \pm 10i \quad (\alpha = 0, \beta = 10)$$

$$x = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

$$\boxed{x = C_1 \cos 10t + C_2 \sin 10t}$$

$x = -\frac{2}{3}$, $t = 0$: $\quad -\frac{2}{3} = C_1$

$$\boxed{x = -\frac{2}{3} \cos 10t + C_2 \sin 10t}$$

→

$$x' = \frac{20}{3}\sin 10t + 10 C_2 \cos 10t$$

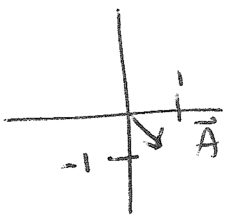
$x'=5$, $t=0$: $5 = 10 C_2$

$$C_2 = \frac{1}{2}$$

$$\boxed{x = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t}$$

b) Write x in the form $A\sin(10t+\phi)$

$$A\sin\phi = -\frac{2}{3}$$
$$A\cos\phi = \frac{1}{2}$$



$\vec{A} = \left[\frac{1}{2}, \frac{-2}{3}\right]$

$$A = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{4}{9} + \frac{1}{4}}$$
$$= \sqrt{\frac{25}{36}}$$
$$= \frac{5}{6}$$

$\phi = \tan^{-1}\left(\frac{A\sin\phi}{A\cos\phi}\right)$ ($+\pi$ if $\vec{A}$ points left)

$$= \tan^{-1}\left(-\frac{2}{3}\cdot\frac{2}{1}\right)$$
$$\approx -0.927$$

$$\boxed{x = \frac{5}{6}\sin(10t - 0.927)}$$

→

$$\text{Amplitude} = A$$
$$= \frac{5}{6}\ m$$

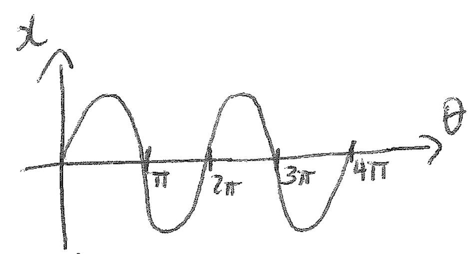
$$\text{Period} = \frac{2\pi}{(10)}$$
$$= \frac{\pi}{5}$$

c) $$\frac{3\pi \ \text{s}}{\left(\frac{\pi}{5}\right) \frac{\text{cycles}}{\text{s}}}$$

$$= 15 \text{ cycles}$$

d) Want $x = 0$, becoming positive

$\uparrow \ominus$
$\downarrow \oplus$



$x = A\sin\theta$

$x = 0$, becoming positive $\Rightarrow$ $\theta = 0, 2\pi, 4\pi, \ldots$

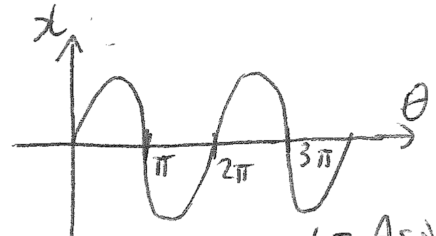
We want $10t - 0.927 = 0, 2\pi, 4\pi$

1st time $10t - 0.927 = 0$
$t = 0.0927$

2nd time $10t - 0.927 = 2\pi$
$t = 0.721\ s$

$\rightarrow$

e) Extreme displacements



$x = A\sin\theta$

Extreme displacements $\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$$= \frac{\pi}{2} + \pi n$$

$$= \frac{\pi}{2} + \frac{2n\pi}{2}$$

$$= \frac{\pi(2n+1)}{2}$$

for $n = 0, 1, 2, \ldots$

We want $10t - 0.927 = \frac{\pi(2n+1)}{2}$

$$10t = \frac{\pi(2n+1)}{2} + 0.927$$

$$t = \frac{\pi(2n+1)}{20} + 0.0927$$

for $n = 0, 1, 2, \ldots$

f) $x(3) = -\frac{2}{3}\cos 30 + \frac{1}{2}\sin 30$

$$\approx -0.597\text{ m}$$

g) velocity $x' = \frac{20}{3}\sin 10t + 5\cos 10t$

$$x'(3) = \frac{20}{3}\sin 30 + 5\cos 30$$

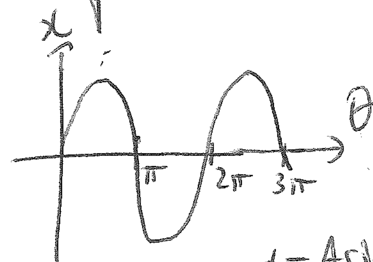
$$\approx -5.816\text{ m/s}$$

→

h) acceleration $x'' = \frac{200}{3}\cos 10t - 50\sin 10t$

$$x''(3) \approx 59.685 \text{ m/s}^2$$

i) times when mass passes through equilibrium



$x = A\sin\theta$

$$x = 0 \Rightarrow \theta = 0, \pi, 2\pi, \dots$$

We want $10t - 0.927 = n\pi$

$$x = \frac{5}{6}\sin(10t - 0.927)$$

$$x' = \frac{50}{6}\cos(10t - 0.927)$$

$$x' = \frac{50}{6}\cos(n\pi) = \pm\frac{50}{6} = \pm\frac{25}{3} \text{ m/s}$$

j) When is $x = \frac{5}{12}$?

$$x = \frac{5}{6}\sin(10t - 0.927)$$

$$\frac{5}{12} = \frac{5}{6}\sin(10t - 0.927)$$

$$\frac{1}{2} = \sin(10t - 0.927)$$

$$\boxed{\begin{aligned} \sin\theta &= \tfrac{1}{2} \\ \Rightarrow \theta &= \tfrac{\pi}{6} + 2\pi n \\ &\text{and } \tfrac{5\pi}{6} + 2\pi n \end{aligned}}$$

$$10t - 0.927 = \frac{\pi}{6} + 2\pi n$$

$$\text{and } 10t - 0.927 = \frac{5\pi}{6} + 2\pi n$$

$$\Rightarrow \quad t = 0.145 + \frac{n\pi}{5}$$

$$\text{and } 0.354 + \frac{n\pi}{5}$$

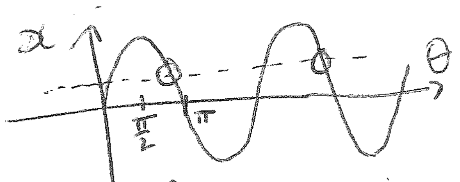
for $n = 0, 1, 2, \ldots$

k) Same as j) but only heading upward

Want $x = \frac{5}{12}$, becoming negative

$\uparrow \ominus$

$\downarrow \oplus$



$x = A\sin\theta$

$$\theta = \frac{5\pi}{6} + 2\pi n$$

$$\Rightarrow t = 0.354 + \frac{n\pi}{5}$$

for $n = 0, 1, 2, \ldots$

(21) $m x'' + \beta x' + kx = f(t)$

m: $\dfrac{1.225\,N}{9.8\,N/kg} = 0.125\,kg$

k: $2\,N/m$

$f(t)$: 0

Initial Conditions

$x(0) = -1\,m$ ↑ ⊖

$x'(0) = 8\,m/s$ ↓ ⊕

$$0.125 x'' + x' + 2x = 0$$

$\times 8$:

$$x'' + 8x' + 16x = 0$$

$$m^2 + 8m + 16 = 0$$

$$(m+4)^2 = 0$$

$$m = -4, -4$$

$$x = (C_1 + C_2 t) e^{-4t}$$

$x = -1$, $t = 0$:

$$-1 = C_1$$

$$x = (-1 + C_2 t) e^{-4t}$$

$$x' = -4(-1 + C_2 t) e^{-4t} + C_2 e^{-4t}$$

$x' = 8$, $t = 0$:

$$8 = 4 + C_2$$

$$C_2 = 4$$

$$\boxed{x = (4t - 1) e^{-4t}}$$

→

Time when $x = 0$:

$$0 = (4t-1)e^{-4t}$$

$$0 = 4t-1$$

$$t = \frac{1}{4} \text{ s}$$

Time when mass attains extreme displacement:

To maximize/minimize x we set $x' = 0$.

$$x' = -4(4t-1)e^{-4t} + 4e^{-4t}$$

$$= (-16t+8)e^{-4t}$$

Set $x' = 0$:

$$0 = (-16t+8)e^{-4t}$$

$$0 = -16t+8$$

$$t = \frac{1}{2} \text{ s}$$

Position of mass at this instant:

$$x\left(\tfrac{1}{2}\right) = e^{-4\left(\frac{1}{2}\right)}$$

$$= e^{-2}$$

$$\approx 0.14 \text{ m}$$

(27) $$mx'' + \beta x' + kx = f(t)$$

m: $\boxed{\dfrac{12.25\text{N}}{9.8\text{ N/kg}} = 1.25\text{ kg}}$

k: $\boxed{\begin{array}{c} 12.25\text{N} = k(2.45\text{m}) \\ k = 5\text{ N/m} \end{array}}$

$f(t)$: 0

$$1.25x'' + \beta x' + 5x = 0$$

$\times 4$: $$5x'' + 4\beta x' + 20x = 0$$

$$5m^2 + 4\beta m + 20 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-4\beta \pm \sqrt{16\beta^2 - 400}}{10}$$

Overdamped $\Rightarrow$ distinct real roots

$\Rightarrow 16\beta^2 - 400 > 0$

$\Rightarrow 16\beta^2 > 400$

$\Rightarrow \beta^2 > 25$ (Recall $\beta \geq 0$)

$\Rightarrow \beta > 5$

$\rightarrow$

Critically damped $\Rightarrow$ repeated real roots

$$\Rightarrow \quad 16\beta^2 - 400 = 0$$

$$\Rightarrow \quad \beta^2 = 25 \qquad \text{(Recall } \beta \geq 0\text{)}$$

$$\Rightarrow \quad \beta = 5$$

Underdamped $\Rightarrow$ complex roots

$$\Rightarrow \quad 16\beta^2 - 400 < 0$$

$$\Rightarrow \quad \beta^2 < 25 \qquad \text{(Recall } \beta \geq 0\text{)}$$

$$\Rightarrow \quad 0 \leq \beta < 5$$

(29) $mx'' + \beta x' + kx = f(t)$

m: $\boxed{\dfrac{4.9\,N}{9.8\,N/kg} = 0.5\,kg}$

β: $\frac{1}{2}$

k: $\boxed{4.9N = k\left(\frac{49}{60}\,m\right) \quad k = 6\ N/m}$

$f(t)$: $10\cos 3t$

Initial Conditions

$x(0) = 2$

$x'(0) = 0$

$\uparrow \ominus$

$\downarrow \oplus$

$$0.5x'' + 0.5x' + 6x = 10\cos 3t$$

$$x'' + x' + 12x = 20\cos 3t$$

1) x_c

$$m^2 + m + 12 = 0$$

$$m = \frac{-1 \pm \sqrt{1-48}}{2}$$

$$m = \frac{-1 \pm \sqrt{47}\,i}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{47}}{2} i \qquad \left(\alpha = \frac{-1}{2},\ \beta = \frac{\sqrt{47}}{2}\right)$$

$$x_c = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

$$= e^{-t/2}\left(c_1\cos\frac{\sqrt{47}}{2}t + c_2\sin\frac{\sqrt{47}}{2}t\right)$$

2) x_p

$$f(t) = 20\cos 3t$$
$$f'(t) = -60\sin 3t$$
$$\vdots$$

$$x_p = A\cos 3t + B\sin 3t$$

3) $x_p \to DE$

$$x_p = A\cos 3t + B\sin 3t$$
$$x_p' = -3A\sin 3t + 3B\cos 3t$$
$$x_p'' = -9A\cos 3t - 9B\sin 3t$$

DE : $\quad x'' + x' + 12x = 20\cos 3t$

$$-9A\cos 3t - 9B\sin 3t + (-3A\sin 3t + 3B\cos 3t) + 12(A\cos 3t + B\sin 3t) = 20\cos 3t$$

$$[\underbrace{-9A + 3B + 12A}_{3A+3B}]\cos 3t + [\underbrace{-9B - 3A + 12B}_{-3A+3B}]\sin 3t = 20\cos 3t + 0\sin 3t$$

$$3A + 3B = 20 \quad (1)$$
$$+ \quad -3A + 3B = 0 \quad (2)$$
$$6B = 20$$
$$B = \tfrac{10}{3} \to (2) \qquad -3A + 10 = 0$$
$$A = \tfrac{10}{3}$$

$$x_p = \tfrac{10}{3}(\cos 3t + \sin 3t)$$

4) $$\boxed{x = e^{-t/2}\left(C_1\cos \tfrac{\sqrt{47}}{2}t + C_2\sin \tfrac{\sqrt{47}}{2}t\right) + \tfrac{10}{3}(\cos 3t + \sin 3t)} \to$$

5) Initial Conditions

$x=2$, $t=0$: $2 = 1\left(C_1\right) + \frac{10}{3}(1)$

$-\frac{4}{3} = C_1$

$$\boxed{x = e^{-t/2}\left(-\frac{4}{3}\cos\frac{\sqrt{47}}{2}t + C_2\sin\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\left(\cos 3t + \sin 3t\right)}$$

$$x' = e^{-t/2}\left(\frac{4}{3}\frac{\sqrt{47}}{2}\sin\frac{\sqrt{47}}{2}t + \frac{\sqrt{47}}{2}C_2\cos\frac{\sqrt{47}}{2}t\right)$$
$$-\frac{1}{2}e^{-t/2}\left(-\frac{4}{3}\cos\frac{\sqrt{47}}{2}t + C_2\sin\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\left(-3\sin 3t + 3\cos 3t\right)$$

$x'=0$, $t=0$: $0 = 1\left(\frac{\sqrt{47}}{2}C_2\right) - \frac{1}{2}\left(-\frac{4}{3}\right) + 10$

$-10 \quad -\frac{2}{3} = \frac{\sqrt{47}}{2}C_2$

$-\frac{32}{3} = \frac{\sqrt{47}}{2}C_2$

$\frac{-64}{3\sqrt{47}} = C_2$

$$x = e^{-t/2}\left(-\frac{4}{3}\cos\frac{\sqrt{47}}{2}t - \frac{64}{3\sqrt{47}}\sin\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\left(\cos 3t + \sin 3t\right)$$