Section 4.7

① $x^2y'' - 2y = 0$

$m(m-1) - 2 = 0$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = -1, 2$

$y = C_1 x^{-1} + C_2 x^2$

③ $xy'' + y' = 0$

$x^2y'' + xy' = 0$

$m(m-1) + m = 0$

$m^2 - m + m = 0$

$m^2 = 0$

$m = 0, 0$

$y_1 = x^0 = 1$

$y_2 = x^0 \ln x = \ln x$

$y = C_1 + C_2 \ln x$

(5) $x^2y'' + xy' + 4y = 0$

$m(m-1) + m + 4 = 0$

$m^2 - m + m + 4 = 0$

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i \quad (\alpha = 0, \beta = 2)$

$y = x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

$y = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)$

(7) $x^2y'' - 3xy' - 2y = 0$

$m(m-1) - 3m - 2 = 0$

$m^2 - m - 3m - 2 = 0$

$m^2 - 4m - 2 = 0$

$m = \frac{4 \pm \sqrt{16 - 4(1)(-2)}}{2}$

$m = \frac{4 \pm \sqrt{24}}{2}$

$m = \frac{4 \pm 2\sqrt{6}}{2}$

$m = 2 \pm \sqrt{6} \qquad y = C_1 x^{2+\sqrt{6}} + C_2 x^{2-\sqrt{6}}$

(9) $25x^2y'' + 25xy' + y = 0$

$25m(m-1) + 25m + 1 = 0$

$25m^2 - 25m + 25m + 1 = 0$

$25m^2 + 1 = 0$

$m^2 = \frac{-1}{25}$

$m = \pm\sqrt{\frac{-1}{25}}$

$m = \pm \frac{1}{5} i \quad (\alpha = 0, \beta = \frac{1}{5})$

$y = x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

$y = C_1 \cos\left(\frac{\ln x}{5}\right) + C_2\left(\frac{\ln x}{5}\right)$

(11) $x^2y'' + 5xy' + 4y = 0$

$m(m-1) + 5m + 4 = 0$

$m^2 - m + 5m + 4 = 0$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m = -2, -2$

$y_1 = x^{-2}$

$y_2 = x^{-2} \ln x$

$y = C_1 x^{-2} + C_2 x^{-2} \ln x$

(13) $3x^2y'' + 6xy' + y = 0$

$$3m(m-1) + 6m + 1 = 0$$

$$3m^2 - 3m + 6m + 1 = 0$$

$$3m^2 + 3m + 1 = 0$$

$$m = \frac{-3 \pm \sqrt{9 - 4(3)(1)}}{6}$$

$$m = \frac{-3 \pm \sqrt{-3}}{6}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}}{6} i \qquad \left(\alpha = \frac{-1}{2},\ \beta = \frac{\sqrt{3}}{6}\right)$$

$$y = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$y = x^{-1/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{6} \ln x\right) + C_2 \sin\left(\frac{\sqrt{3}}{6} \ln x\right)\right]$$

(15) $x^3 y''' - 6y = 0$

$m(m-1)(m-2) - 6 = 0$

$(m^2 - m)(m-2) - 6 = 0$

$m^3 - 2m^2 - m^2 + 2m - 6 = 0$

$m^3 - 3m^2 + 2m - 6 = 0$

Rational Roots Theorem:

Possibilities for p: $\pm 1, \pm 2, \pm 3, \pm 6$

" q: ± 1

" $\frac{p}{q}$: $\underbrace{\pm 1, \pm 2, \pm 3, \pm 6}_{\text{These are the possible roots}}$

Check $m=-1$: $(-1)^3 - 3(-1)^2 + 2(-1) - 6 = 0$ ✗

$m=1$: $1^3 - 3 + 2 - 6 = 0$ ✗

$m=-2$: $(-2)^3 - 3(-2)^2 + 2(-2) - 6 = 0$ ✗

$m=2$: $2^3 - 3(4) + 4 - 6 = 0$ ✗

$m=-3$: $(-3)^3 - 3(-3)^2 + 2(-3) - 6 = 0$ ✗

$m=3$: $3^3 - 3(3^2) + 2(3) - 6 = 0$ ✓

$m=3$ is a root $\Rightarrow$ $m-3$ is a factor

of $m^3 - 3m^2 + 2m - 6$

→

$$
\begin{array}{r|l}
 & m^2+2 \\
\hline
m-3 & m^3-3m^2+2m-6 \\
 & -(m^3-3m^2) \\
\hline
 & \qquad 2m-6 \\
 & \qquad -(2m-6) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$m^3-3m^2+2m-6=0$$

$$(m-3)(m^2+2)=0$$

$$m=3 \qquad m^2=-2$$

$$m=\pm\sqrt{-2}$$

$$m=\pm\sqrt{2}\,i \quad (\alpha=0,\ \beta=\sqrt{2})$$

$$y=C_1x^3+C_2\cos(\sqrt{2}\ln x)+C_3\sin(\sqrt{2}\ln x)$$

(17) $xy^{(4)} + 6y''' = 0$

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) = 0$$

Factor!

$$m(m-1)(m-2)[m-3+6] = 0$$

$$m(m-1)(m-2)(m+3) = 0$$

$$m = 0, 1, 2, -3$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^{-3}$$

(19) $xy'' - 4y' = x^4$

$$x^2 y'' - 4xy' = x^5$$

y_c: $m(m-1) - 4m = 0$

$$m^2 - m - 4m = 0$$

$$m^2 - 5m = 0$$

$$m(m-5) = 0$$

$$m = 0, 5$$

$$y_c = C_1 + C_2 x^5$$

$\rightarrow$

Standard form $y'' - \frac{4}{x}y' = x^3 \leftarrow f(x)$

$$W = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$$

$$W_1 = \begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix} = -x^8$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix} = x^3$$

$$u_1' = \frac{W_1}{W} \qquad\qquad u_2' = \frac{W_2}{W}$$

$$= \frac{-x^4}{5} \qquad\qquad = \frac{1}{5}\frac{1}{x}$$

$$u_1 = \frac{-x^5}{25} \qquad\qquad u_2 = \frac{1}{5}\ln|x|$$

$$y_p = u_1y_1 + u_2y_2$$

$$y_p = \frac{-x^5}{25} + \frac{x^5}{5}\ln|x|$$

Now $y = y_c + y_p$

$$y = C_1 + \underbrace{C_2x^5 - \frac{x^5}{25}} + \frac{x^5}{5}\ln|x|$$

or $y = C_1 + C_3x^5 + \frac{x^5}{5}\ln|x|$

or $y = C_1 + C_3x^5 + \frac{x^5}{5}\ln x \qquad (x > 0)$

(21) $x^2y'' - xy' + y = 2x$

y_c: $m(m-1) - m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y_c = C_1 x + C_2 x \ln x$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x(1+\ln x) - x\ln x = x$$

$x\left(\frac{1}{x}\right) + \ln x$

Standard ~~Form~~: $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$

$f(x)$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & 1+\ln x \end{vmatrix} = -2\ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix}$$

$= 2$

$u_1' = \frac{W_1}{W}$

$= \frac{-2\ln x}{x}$

$u_2' = \frac{W_2}{W}$

$= \frac{2}{x}$

$\rightarrow$

$$u_1' = \frac{-2\ln x}{x} \qquad u_2' = \frac{2}{x}$$

$$u_1 = \int -\frac{2\ln x}{x}\,dx \qquad \boxed{\text{Sub } z = \ln x}$$

$$= \int -2z\,dz$$

$$= -z^2$$

$$= -(\ln x)^2$$

$$u_2 = 2\ln x \qquad (x > 0)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -x(\ln x)^2 + (2\ln x)\,x\ln x$$

$$= -x(\ln x)^2 + 2x(\ln x)^2$$

$$= x(\ln x)^2$$

$$y = C_1 x + C_2 x\ln x + x(\ln x)^2$$

(23) $x^2y'' + xy' - y = \ln x$

y_c: $m(m-1) + m - 1 = 0$

$m^2 - m + m - 1 = 0$

$m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m = -1, 1$

$y_c = C_1 x^{-1} + C_2 x$

$$W = \begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix} = x^{-1} + x^{-1} = 2x^{-1} = \frac{2}{x}$$

Standard form: $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \underbrace{\frac{\ln x}{x^2}}_{f(x)}$

$$W_1 = \begin{vmatrix} 0 & x \\ \frac{\ln x}{x^2} & 1 \end{vmatrix} = -\frac{\ln x}{x}$$

$$W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{\ln x}{x^2} \end{vmatrix} = \frac{\ln x}{x^3}$$

$$u_1' = \frac{W_1}{W} = \frac{-\ln x}{x}\left(\frac{x}{2}\right) = \frac{-\ln x}{2}$$

$$u_2' = \frac{W_2}{W} = \frac{\ln x}{x^3}\left(\frac{x}{2}\right) = \frac{\ln x}{2x^2} \rightarrow$$

$$u_1' = \frac{-\ln x}{2}$$

D		I
$\ln x$	$+$	$-\frac{1}{2}$
$\frac{1}{x}$	$-$	$-\frac{x}{2}$

$$u_1 = -\frac{x}{2}\ln x + \int \frac{1}{2}\,dx$$

$$= -\frac{x}{2}\ln x + \frac{x}{2}$$

$$u_2' = \frac{\ln x}{2x^2}$$

D		I
$\ln x$	$+$	$\frac{1}{2}x^{-2}$
$\frac{1}{x}$	$-$	$-\frac{1}{2}x^{-1}$

$$u_2 = -\frac{1}{2}x^{-1}\ln x + \int \frac{1}{2}x^{-2}\,dx$$

$$= -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{x}{2}\ln x + \frac{x}{2}\right)x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x$$

$$= -\frac{1}{2}\ln x + \frac{1}{2} - \frac{1}{2}\ln x - \frac{1}{2}$$

$$= -\ln x$$

$$y = y_c + y_p$$

$$y = C_1 x^{-1} + C_2 x - \ln x$$

(25) $x^2y'' + 3xy' = 0$, $y(1) = 0$, $y'(1) = 4$

$m(m-1) + 3m = 0$

$m^2 - m + 3m = 0$

$m^2 + 2m = 0$

$m(m+2) = 0$

$m = 0, -2$

$$\boxed{y = C_1 + C_2 x^{-2}}$$

$y = 0$
$x = 1$: $0 = C_1 + C_2$ (1)

$y' = -2C_2 x^{-3}$

$y' = 4$
$x = 1$: $4 = -2C_2 \Rightarrow C_2 = -2$

$C_2 = -2 \rightarrow$ (1): $C_1 - 2 = 0$

$C_1 = 2$

$$\boxed{y = 2 - 2x^{-2}}$$

(27) $x^2y'' + xy' + y = 0, \quad y(1)=1, \quad y'(1)=2$

$$m(m-1) + m + 1 = 0$$
$$m^2 - m + m + 1 = 0$$
$$m^2 + 1 = 0$$
$$m^2 = -1$$
$$m = \pm\sqrt{-1}$$
$$m = \pm i \quad (\alpha=0, \beta=1)$$

$$y = x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$$
$$\boxed{y = C_1 \cos(\ln x) + C_2 \sin(\ln x)}$$

$y=1$, $x=1$: $\quad 1 = C_1(\cos 0) + C_2(\sin 0)$

$$1 = C_1$$
$$\boxed{y = \cos(\ln x) + C_2 \sin(\ln x)}$$

$$y' = -\sin(\ln x)\frac{1}{x} + C_2 \cos(\ln x)\frac{1}{x}$$

$y'=2$, $x=1$: $\quad 2 = -(\sin 0)(1) + C_2(\cos 0)(1)$

$$2 = C_2$$

$$y = \cos(\ln x) + 2\sin(\ln x)$$

(29) $xy'' + y' = x \ , \ y(1) = 1 \ , \ y'(1) = -\frac{1}{2}$

$x^2 y'' + x y' = x^2$

y_c: $m(m-1) + m = 0$

$m^2 - m + m = 0$

$m^2 = 0$

$m = 0, 0 \qquad y_1 = x^0 = 1 \quad y_2 = \ln x$

$y_c = C_1 + C_2 \ln x$

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$

standard form: $y'' + \frac{1}{x} y' = \underset{f(x)}{1}$

$$W_1 = \begin{vmatrix} 0 & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = -\ln x$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$u_1' = \frac{W_1}{W} \qquad\qquad u_2' = \frac{W_2}{W}$

$= -x \ln x \qquad\qquad = x$

$\rightarrow$

$u_1' = -x\ln x$

$u_2' = x$

$u_2 = \frac{x^2}{2}$

D		I
$\ln x$	$+$	$-x$
$\frac{1}{x}$	$-$	$-\frac{x^2}{2}$

$$u_1 = -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4} + \frac{x^2}{2}\ln x$$

$$= \frac{x^2}{4}$$

$$\boxed{y = C_1 + C_2 \ln x + \frac{x^2}{4}}$$

$y = 1$, $x = 1$: $1 = C_1 + \frac{1}{4} \Rightarrow C_1 = \frac{3}{4}$

$$\boxed{y = \frac{3}{4} + C_2 \ln x + \frac{x^2}{4}}$$

$$y' = \frac{C_2}{x} + \frac{x}{2}$$

$y' = -\frac{1}{2}$, $x = 1$: $-\frac{1}{2} = C_2 + \frac{1}{2} \Rightarrow C_2 = -1$

$$\boxed{y = \frac{3}{4} - \ln x + \frac{x^2}{4}}$$